The Angular-Momentum Spectrum of Kerr Black Holes: Basis of the Existence Criterion, Spin Evolution and Breaking

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Abstract

In the capsule-counting framework, the energy of a black hole is quantized in whole packs of the dark-energy capsule E_c, while angular momentum has no independent quantum and is derived from the two horizon entropies. A black-hole state is fixed by two integers, the total capsule number N and the irreducible capsule number N_{irr}; the angular momentum, $J = \hbar\sqrt{S_+S_-}/(2\pi k_B)$, is proportional to the geometric mean of the two entropies, and a capsule entering or leaving is registered as a whole pack in either the irreducible or the rotational part. By the $2d_m$ axiom, the lower limit $2\ell_p$ on the horizon separation is the minimum distinguishable span, of the same origin as the coefficient 1/4 of the Bekenstein–Hawking entropy. This yields a two-level spectrum: whole-pack states with $N \leq 4$ cannot rotate, the minimal rotating state is $BH(5, 4)$, and the J of whole-pack states is in general not an integer multiple of $\hbar$; external exchange of angular momentum takes place at integer sub-step states (ab a perfect square), where $J = 2\hbar\sqrt{ab}$ is an integer multiple of $2\hbar$, every even multiple from $8\hbar$ upward is reachable, and this is compatible with integer quantization. $N = 5$ is both the first stable structure and the minimal rotating state, and small black holes grow in a way structurally analogous to Ostwald's rule of stages. For large N, probabilistic choice between the two registrations recovers Bardeen's spin-evolution equation on average. Only irreducible capsules are emitted, so rotational capsules must first be re-registered; the first law, $T\,\Delta S_+ \simeq \Omega_H(L - \Delta J_{rot})$, fixes the average number of re-registrations $\bar{m}$, and above a threshold spin $\chi_c \approx 0.88$–0.92 the outer-horizon area increases on average, locating Page's result. The spin limit $\chi_{max} = 1-1/N^2$ is the existence boundary, with horizon separation exactly $2\ell_p$; near it absorbing one rotational pack causes breaking in a single step, before the classical extremal point and before the temperature reaches zero. The framework agrees on average with general relativity and semiclassical results and gives discrete structure at small N and at the spin limit.

Keywords: Black-hole angular-momentum spectrum, Existence criterion, Kerr black hole, Small black holes, Spin evolution, Hawking radiation, Black-hole breaking

1 Introduction

A Kerr black hole is completely determined by two quantities, its mass and its angular momentum [1], and its entropy is proportional to the area of its outer horizon, $S = k_B A/(4\ell_p^2)$ [2, 3]. In general relativity both mass and angular momentum are continuous, and the evolution of spin is well established [4]: thin-disk accretion raises the spin [5, 6], disk radiation stops the spin at about $\chi = 0.998$ [7], Hawking radiation lowers the spin, and when the spin is high enough the outer-horizon area first increases [8].

The previous papers of this series start from the $2d_m$ axiom and give a different description. Paper I [9] introduced the dark-energy capsule E_c of energy E_P. Paper II [10] counted black-hole entropy by the capsule number N: $S/k_B = 4\pi N^2$ is exactly the sum of the self terms of all capsules and their ordered pairwise relations. For a Kerr black hole, the outer horizon registers the irreducible part and the inner horizon the rotational part together with the cross relations; the separation of the two horizons being not smaller than $2\ell_p$ is the existence criterion, which gives the highest allowed spin $\chi_{max} = 1 - 1/N^2$. Supermassive black holes that satisfy the criterion and go on to breaking can be regarded as "cosmic egg cells" that give birth to the next generation of universes; this idea is left to later work.

In this framework the energy of a black hole enters and leaves in whole packs of E_P, yet angular momentum has no independent quantum and can only be derived from the entropies of the two horizons. This raises the questions that this paper answers: where angular momentum comes from, how it is gained and lost, and where it ends. Specifically there are four:

(1) Why is the lower limit of the existence criterion $2\ell_p$? Paper II gave only an intuitive picture.
(2) Which angular momenta does the criterion allow? What does the structure of the spectrum look like at small N? How is the externally visible angular momentum quantized?
(3) How does angular momentum change when capsules enter and leave in whole packs? Can the continuous results of general relativity and semiclassical calculations, namely Bardeen's spin evolution, the Thorne limit and Page's evaporation results, be recovered on average?
(4) How does the spin go from the Thorne limit to the upper limit, and how does breaking occur at the limit?

The main idea of this paper consists of two rules:

- **Distinguishability rule**: the two registrations must be separated by the minimum distinguishable span $2\ell_p$. It determines the angular-momentum spectrum and also the end point of the spectrum, beyond which the black hole breaks.

- **Whole-pack registration rule**: each capsule can only be registered, as a whole pack, either in the irreducible part or in the rotational part. It makes each step of change in angular momentum take only two discrete values.

The known continuous results should be the average over a large number of capsules registered probabilistically; the difference between the value registered for a single capsule and the angular momentum actually brought in or carried out appears as the exchange of angular momentum between the black hole and its exterior, and as re-registration during evaporation; what is emitted is always an irreducible capsule, a rotational capsule must first be re-registered, and re-registration is a thermodynamic tendency whose rate is controlled by the radiation. At small N and at the spin limit, averaging no longer applies and the discrete structure shows directly. Energy is quantized in whole packs, while the externally visible change in angular momentum is referred to the sub-steps, where it comes in units of $2\hbar$.

Section 2 lists the results carried over from the previous papers. Section 3 argues for the lower limit $2\ell_p$ from the $2d_m$ axiom, gives the structure of the angular-momentum spectrum and the quantization of angular momentum on sub-steps, and compares them with the integer quantization rule. Section 4 discusses the structure, growth and evaporation of small-N black holes. Section 5 gives the registration rules and discusses the gain and loss of angular momentum in accretion and evaporation. Section 6 discusses how large-N black holes approach the upper limit and break. Section 7 gives the conclusions and Section 8 lists open questions. Throughout, we keep as close as possible to established theory: the results of general relativity and semiclassical calculations are quoted directly, and on this basis the framework gives the results of capsule counting, supplementing them with discrete structure that is invisible in continuous theory.

2 Results carried over from the previous papers

This paper quotes results already obtained in the previous papers without repeating their derivations. The results needed are collected in this section.

Paper I [9]. The dark-energy capsule E_c has inner diameter $2\ell_p$, outer diameter $4\ell_p$ and energy $E_P = m_P c^2$. The capsule density is denoted ρ_c. A black hole with the energy of N capsules has mean density ρ_c/N^2 (Eq. (31) of Paper I).

Paper II [10]. For a black hole containing N capsules, $E = NE_P$, $M = Nm_P$, and

$$R_s = \frac{2GM}{c^2} = \frac{2GNm_P}{c^2} = 2N\ell_p, \qquad \frac{S}{k_B} = \frac{4\pi R_s^2}{4\ell_p^2} = 4\pi N^2, \qquad k_B T_H = \frac{E_P}{8\pi N}, \quad (1)$$

where $Gm_P/c^2 = \ell_p$ has been used. Accretion and evaporation are quantized in whole packs of E_P, last in, first out. In packing $N \to N+1$ and unpacking $N \to N-1$, the black hole absorbs or emits energy E_P and entropy $4\pi k_B(2N \pm 1)$; the corresponding

temperatures are

$$k_B T_N^{\pm} = \frac{E_P}{4\pi(2N \pm 1)}, \qquad \frac{1}{T_N^+} + \frac{1}{T_N^-} = \frac{2}{T_H}. \tag{2}$$

T_H is the continuous value at N and also the harmonic mean of T_N^+ and T_N^-; T_N^- equals the value of T_H at $N - \frac{1}{2}$. At large N the two agree, while at small N whole-pack entry and exit require $T_N^{\pm}$. The smallest black hole forms from two capsules in contact, $1 + 1 \to 2 \to 1$. Kerr black hole: the irreducible mass is $M_{irr} = N_{irr} m_P$ and the rotational part is $N_{rot} = N - N_{irr}$; the entropies of the two horizons are

$$\frac{S_+}{k_B} = 4\pi N_{irr}^2, \qquad \frac{S_-}{k_B} = 4\pi\left(N^2 - N_{irr}^2\right) = 4\pi N_{rot}^2 + 8\pi N_{irr} N_{rot}. \tag{3}$$

The Kerr horizon equation $r^2 - R_s r + a^2 = 0$ can be written as $(r - R_i)^2 + a^2 = R_i^2$ ($R_i = R_s/2$): in the (r, a) plane it is a circle centred at $(R_i, 0)$ with diameter $OB = R_s$, called the horizon circle (Paper II [10], Fig. 3). The outer horizon corresponds to a point $P = (r_+, a)$ on the circle. By Thales' theorem $\angle OPB$ is a right angle, $OP = \sqrt{r_+^2 + a^2} = \sqrt{R_s r_+} = R_{A+}$ and $PB = \sqrt{r_-^2 + a^2} = R_{A-}$. Denoting the inscribed angle $\angle POB = \omega$ (central angle 2ω),

$$\begin{aligned} &R_{A+} = R_s \cos\omega = 2N_{irr}\ell_p, \qquad R_{A-} = R_s \sin\omega = 2\ell_p\sqrt{N^2 - N_{irr}^2}, \\ &R_{A+}^2 + R_{A-}^2 = R_s^2. \end{aligned} \tag{4}$$

From $R_{A+} = 2N_{irr}\ell_p$ and $R_s = 2N\ell_p$, $\cos\omega = N_{irr}/N = M_{irr}/M$.

The effective count of the inner-horizon areal radius in units of $2\ell_p$ is used in several places below; we denote it

$$k \equiv \frac{R_{A-}}{2\ell_p} = \sqrt{N^2 - N_{irr}^2}. \tag{5}$$

In units of $2\ell_p$, the right triangle on the horizon circle has hypotenuse N and legs N_{irr} and k; this is the form used for counting in this paper. A black-hole state is labelled $BH(N, N_{irr}, N_{rot}, k)$; since $N_{rot} = N - N_{irr}$ and $k = \sqrt{N^2 - N_{irr}^2}$ are both fixed by the first two numbers, the state is also written briefly as $BH(N, N_{irr})$, and tables give only the pair (N, N_{irr}). For example, $BH(5, 4, 1, 3)$ denotes $N = 5$, $N_{irr} = 4$, $N_{rot} = 1$, $k = 3$, written briefly as $BH(5, 4)$.

From Eqs. (3) and (5), $S_-/k_B = 4\pi k^2$. In general k is not an integer (Section 3.7).

$$\chi = \frac{a}{GM/c^2} = \sin 2\omega, \qquad \mathcal{J} \equiv \frac{J}{\hbar} = \chi N^2. \tag{6}$$

The existence criterion and the allowed window:

$$R_{A+} - R_{A-} \geq 2\ell_p, \qquad \chi \leq \chi_{max} = 1 - \frac{1}{N^2}, \qquad \frac{\sqrt{2}}{2} < \frac{N_{irr}}{N} \leq 1. \tag{7}$$

Zero separation ($R_{A+} = R_{A-}$) is the classical extremal point, which cannot be reached; a separation below $2\ell_p$ means breaking. The χ_{max} of Eq. (7) is a continuous upper limit; at small N the actual highest rotating spin of each N lies below it, and χ_{max} is their envelope (Section 3.8). Black holes with $N \leq 4$ cannot rotate, and the minimal rotating state is $BH(N, N_{irr}) = BH(5, 4)$. The Kerr temperature is

$$k_B T = \frac{E_P}{8\pi N} \left(1 - \tan^2 \omega\right). \tag{8}$$

3 Basis of the existence criterion and structure of the angular-momentum spectrum

This section first gives the basis of the lower limit $2\ell_p$ from the $2d_m$ axiom (Section 3.1) and then the angular-momentum spectrum allowed by the criterion. Paper II [10] gave the boundaries of the spectrum: the upper limit χ_{max} and the minimal rotating state $BH(5, 4)$; Sections 3.2–3.7 give the full picture of the spectrum.

3.1 Basis of the lower limit $2\ell_p$

Paper II [10] used the picture "the two horizons each have thickness ℓ_p, separated by a further ℓ_p" to explain why the lower limit on the separation of the centres of the two horizon interfaces is $2\ell_p$. This is only an intuitive picture, in which "thickness ℓ_p" and "separation ℓ_p" are each a model assumption. In this section the minimum distinguishable span is obtained from the $2d_m$ axiom.

The starting point of Paper I [9] is that information is the measure of distinguishable relations between objects. A relation requires at least two minimal units, so the minimum distinguishable span is $2d_m = 2\ell_p$. The coefficient $1/4$ of black-hole entropy originates exactly here: the minimal area element has side $2\ell_p$, and the area $(2\ell_p)^2 = 4\ell_p^2$ carries one relation,

$$\frac{S}{k_B} = \frac{A}{(2\ell_p)^2} = \frac{A}{4\ell_p^2}. \tag{9}$$

The two horizons of a rotating black hole register two different pieces of information: the outer horizon registers the irreducible part $4\pi k_B N_{irr}^2$, and the inner horizon the rotational part together with the cross relations, $4\pi k_B (N_{rot}^2 + 2N_{irr}N_{rot})$. Relations can be counted only if they can be distinguished. For the two registrations to remain distinguishable, they must be separated in the radial direction by at least one minimum distinguishable span:

$$g \equiv R_{A+} - R_{A-} \geq 2d_m = 2\ell_p. \tag{10}$$

If the two horizons come closer, the two layers of registration units overlap and cannot be distinguished; this is what Paper II [10] described as "the registrations mix and can no longer be distinguished". Internally, $2d_m$ is the minimal span required for one relation to exist, i.e. the inner diameter of a capsule; externally, it is the minimal separation at which two relations can be distinguished, i.e. $2\ell_p$. The lower limit $2\ell_p$ and the coefficient $1/4$ come from the same axiom, and this is an application of the $2d_m$ axiom to the two layers of registration of the two horizons, introducing no new parameter.

3.2 Counting form of the criterion

Substituting Eq. (4) into criterion Eq. (10):

$$2N_{irr}\ell_p - 2\ell_p\sqrt{N^2 - N_{irr}^2} \geq 2\ell_p. \tag{11}$$

Dividing both sides by $2\ell_p$ and using k from Eq. (5):

$$N_{irr} - k \geq 1. \tag{12}$$

That is, the radial counts differ by at least one whole unit. Since $N_{irr} - 1 \geq k \geq 0$, squaring both sides gives $(N_{irr} - 1)^2 \geq k^2$, i.e.

$$N_{irr}^2 - 2N_{irr} + 1 \geq N^2 - N_{irr}^2 \quad \Rightarrow \quad 2N_{irr}^2 - 2N_{irr} + \left(1 - N^2\right) \geq 0. \tag{13}$$

This is a quadratic inequality in N_{irr}, with roots

$$N_{irr} = \frac{2 \pm \sqrt{4 - 8\,(1 - N^2)}}{4} = \frac{1 \pm \sqrt{2N^2 - 1}}{2}. \tag{14}$$

Since N_{irr} is positive, the criterion is equivalent to

$$N_{irr} \geq \left\lceil \frac{1 + \sqrt{2N^2 - 1}}{2} \right\rceil = N_{irr,min}. \tag{15}$$

The condition for the lowest rotating state, $N_{rot} = 1$, $N_{irr} = N - 1$, to be allowed follows by substituting $N_{irr} = N - 1$ into Eq. (13):

$$2(N-1)^2 - 2(N-1) + 1 - N^2 = N^2 - 6N + 5 = (N-1)(N-5) \geq 0, \tag{16}$$

so black holes with $N \leq 4$ cannot rotate; at $N = 5$ the equality holds and $BH(5, 4)$ lies exactly on the lower limit, in agreement with Paper II [10].

3.3 Number of rotating states for each N

A rotating state requires at least one capsule to belong to the rotational part, i.e. $N_{irr} \leq N-1$; so for given N the allowed rotating states N_{irr} form a run of consecutive

integers:

$$N_{irr,min} \le N_{irr} \le N-1.$$

The number of consecutive integers is $n_{rot}(N) = (N-1) - N_{irr,min} + 1 = N - N_{irr,min}$. With Eq. (15), the number of rotating states for N is

$$n_{rot}(N) = N - \left\lceil \frac{1+\sqrt{2N^2-1}}{2} \right\rceil . \tag{17}$$

At large N, $\sqrt{2N^2-1} \approx \sqrt{2}\,N$, and

$$n_{rot}(N) \approx N\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{2} \approx 0.29289N - 0.5. \tag{18}$$

Note that n_{rot} is the number of rotating states, not the number of rotational capsules N_{rot} of a given state. The two are related numerically: the highest-spin state takes $N_{irr} = N_{irr,min}$, and its number of rotational capsules is $N_{rot,max} = N - N_{irr,min}$; since N_{irr} runs from $N_{irr,min}$ to $N-1$ without gaps, the rotating states have $N_{rot} = 1, 2, \ldots, N_{rot,max}$ rotational capsules in turn, so $n_{rot}(N) = N_{rot,max}$. For example, for $N = 12$ the allowed values are $N_{irr} = 11, 10, 9$, three rotating states with 1, 2 and 3 rotational capsules in turn, and the highest-spin state $BH(12,9)$ also has 3 rotational capsules.

For $N = 10$, 100 and 1000, Eq. (17) before rounding up gives 2.44, 28.79 and 292.39, and Eq. (18) gives 2.43, 28.79 and 292.39, so the two differ very little; after rounding up, Eq. (17) gives 2, 28 and 292 rotating states. The number of rotating states first reaches 1, 2, 3, 4, 5, 6 at $N = 5$, 9, 12, 16, 19, 23 respectively.

3.4 Counting form of angular momentum and spin

From the right triangle on the horizon circle, $\cos\omega = N_{irr}/N$ and $\sin\omega = k/N$, and from Eq. (6):

$$\chi = \sin 2\omega = 2\sin\omega\cos\omega = \frac{2N_{irr}k}{N^2}, \tag{19}$$

$$\mathcal{J} = \chi N^2 = \frac{2N_{irr}k}{N^2}\,N^2 = 2N_{irr}k = 2N_{irr}\sqrt{N^2 - N_{irr}^2}. \tag{20}$$

The same result follows directly from the Christodoulou mass formula:

$$M^2 = M_{irr}^2 + \frac{J^2c^2}{4G^2M_{irr}^2}. \tag{21}$$

Solving for J:

$$J^2 = \frac{4G^2M_{irr}^2\left(M^2 - M_{irr}^2\right)}{c^2}, \qquad J = \frac{2G}{c}\,M_{irr}\sqrt{M^2 - M_{irr}^2}.$$

Substituting $M = Nm_P$ and $M_{irr} = N_{irr}m_P$, and using $Gm_P^2/c = \hbar$:

$$J = \frac{2Gm_P^2}{c} N_{irr}\sqrt{N^2 - N_{irr}^2} = 2\hbar\, N_{irr}\sqrt{N^2 - N_{irr}^2}. \tag{22}$$

This is the dimensional angular momentum J; Eq. (20) gives the dimensionless angular momentum in units of $\hbar$, $\mathcal{J} = J/\hbar$ (Eq. (6)), and the two differ by a factor $\hbar$. Dividing both sides of Eq. (22) by $\hbar$:

$$\mathcal{J} = \frac{J}{\hbar} = 2N_{irr}\sqrt{N^2 - N_{irr}^2},$$

which is the same as Eq. (20). What is quantized in this framework is energy: the total energy and the irreducible energy of a black hole are both integer multiples of E_P, i.e. N and N_{irr} are both integers. Angular momentum is not quantized on its own but is computed from these two integers by the Christodoulou mass formula, so J can also take only discrete values. The expression, however, contains the square root $\sqrt{N^2 - N_{irr}^2}$, and these discrete values are in general not integer multiples of $\hbar$ (Section 3.7).

3.5 Allowed states for $N = 5$–13

Table 1 lists all rotating states for $N = 5$–13 (lengths in units of ℓ_p; $\mathcal{J} = J/\hbar$ see Eq. (6), g see Eq. (10)). Non-rotating states ($N_{irr} = N$) are not listed. $R_{A-} - R_i > 0$, i.e. $\chi > \sqrt{3}/2$, is the eukaryotic type; this division coincides with the critical value of Smarr's horizon embedding [11]. The boundary $R_{A-} = R_i$ corresponds to $\sin\omega = 1/2$, i.e. $\cos\omega = \sqrt{3}/2$; since $\cos\omega = N_{irr}/N$ is rational while $\sqrt{3}/2$ is irrational, no black-hole state lies exactly on the prokaryotic/eukaryotic boundary, and $R_{A-} - R_i$ is nonzero for every entry of Table 1. The boundary is nevertheless no barrier: the existence criterion constrains only the separation of the two horizons and does not involve R_i, so a black-hole state can cross the boundary in a single step with each pack it gains or loses; for example, $BH(7,6)$ (eukaryotic) absorbs an irreducible pack and becomes $BH(8,7)$ (prokaryotic). The inner horizon is a registration interface, not a material wall; E_c capsules contribute to gravity only through their residual (Paper I [9]) and present no obstruction to the registration interface or to other bodies, so when R_{A-} crosses R_i it is neither obstructed by the capsules nor disturbs their placement. The stability of the inner horizon as a Cauchy horizon is discussed in Paper II [10], Section 8.6: the Cauchy horizon of a Kerr black hole is C^0-stable, and the inner horizon does not disappear as a surface.

The excluded states all lie at the high-spin end of each N, with separation below $2\ell_p$: $BH(4,3)$ 0.71, $BH(7,5)$ 0.20, $BH(8,6)$ 1.42, $BH(11,8)$ 0.90 (in units of ℓ_p).

From Table 1 one can read:

1. $N = 5$, 6 and 7 each have only one rotating state, $N_{irr} = N-1$, all in the eukaryotic region; $R_{A-} - R_i$ is 1.00, 0.63 and 0.21 ℓ_p in turn, so the gap between the inner horizon and the inner interface narrows step by step.

Table 1 Rotating states for $N = 5$–13

N	N_{irr}	N_{rot}	$\mathcal{J}$	χ	g/ℓ_p	R_{A+}/ℓ_p	R_{A-}/ℓ_p	$(R_{A-} - R_i)/\ell_p$	Type
5	4	1	24	0.960	2.00	8	6.00	+1.00	eukaryotic
6	5	1	33.17	0.921	3.37	10	6.63	+0.63	eukaryotic
7	6	1	43.27	0.883	4.79	12	7.21	+0.21	eukaryotic
8	7	1	54.22	0.847	6.25	14	7.75	−0.25	prokaryotic
9	8	1	65.97	0.814	7.75	16	8.25	−0.75	prokaryotic
9	7	2	79.20	0.978	2.69	14	11.31	+2.31	eukaryotic
10	9	1	78.46	0.785	9.28	18	8.72	−1.28	prokaryotic
10	8	2	96	0.960	4.00	16	12.00	+2.00	eukaryotic
11	10	1	91.65	0.758	10.84	20	9.17	−1.83	prokaryotic
11	9	2	113.84	0.941	5.35	18	12.65	+1.65	eukaryotic
12	11	1	105.51	0.733	12.41	22	9.59	−2.41	prokaryotic
12	10	2	132.66	0.921	6.73	20	13.27	+1.27	eukaryotic
12	9	3	142.87	0.992	2.12	18	15.87	+3.87	eukaryotic
13	12	1	120	0.710	14.00	24	10.00	−3.00	prokaryotic
13	11	2	152.42	0.902	8.14	22	13.86	+0.86	eukaryotic
13	10	3	166.13	0.983	3.39	20	16.61	+3.61	eukaryotic

2. Prokaryotic rotating states appear from $N = 8$: the inner horizon of $BH(8, 7)$ falls back inside R_i for the first time. $N = 8$ still has only one rotating state; $BH(8, 6)$ is excluded because its separation $1.42\ell_p < 2\ell_p$.
3. A second rotating state appears from $N = 9$, with prokaryotic and eukaryotic types coexisting; a third appears from $N = 12$.
4. $BH(10, 8)$ has the same spin as $BH(5, 4)$ ($\chi = 0.960$); the former is a twofold enlargement of the horizon triangle of the latter.

At small N, once a black hole rotates, its spin is large. A rotating state has at least one rotational capsule ($N_{rot} \geq 1$), and this capsule makes up a large share of N, so even the lowest rotating spin is high ($\chi = 0.960$ for $N = 5$). The criterion is the same as at large N; only the discreteness becomes visible.

3.6 Angular momentum on sub-steps

Energy is quantized in whole packs E_P, and N and N_{irr} change only when a pack is completed. The whole-pack counting inside is not visible from outside; only the sub-steps are: the entry or exit of each pack is divided into $2N \pm 1$ sub-steps, each bringing in or carrying out entropy $4\pi k_B$ (Section 2; Paper II). The external change of angular momentum should therefore be referred to the sub-steps. This section gives the angular momentum on sub-steps and its quantization condition.

Sub-step counting. In units of sub-steps, the relation counts of the two horizons are

$$a \equiv \frac{S_+}{4\pi k_B}, \qquad b \equiv \frac{S_-}{4\pi k_B}, \qquad a,\ b \in \mathbb{N}. \tag{23}$$

Each sub-step changes a or b by 1. When a pack is completed, by Eq. (3), $a = N_{irr}^2$, $b = k^2 = N^2 - N_{irr}^2$ and $a + b = N^2$; between packs, a and b change sub-step by sub-step and need not be perfect squares.

Angular momentum on sub-steps. By Eq. (41), $J = \hbar\sqrt{S_+ S_-}/(2\pi k_B)$. Substituting $S_+ = 4\pi k_B a$ and $S_- = 4\pi k_B b$:

$$J = \frac{\hbar}{2\pi k_B}\sqrt{(4\pi k_B a)(4\pi k_B b)} = \frac{\hbar}{2\pi k_B} \cdot 4\pi k_B \sqrt{ab} = 2\hbar\sqrt{ab}. \tag{24}$$

When a pack is completed, $\sqrt{ab} = \sqrt{N_{irr}^2 k^2} = N_{irr} k$, and Eq. (24) reduces to Eq. (42), $J = 2\hbar N_{irr} k$.

Black-hole states and sub-step states. The state registered after a whole pack is completed is called a black-hole state and is written $BH(N, N_{irr})$; the existence criterion, Eq. (12), is checked only when a black-hole state is registered. A black-hole state absorbs or emits one share E_P and reaches the next black-hole state through $2N \pm 1$ sub-steps; in accretion each sub-step increases $a + b$ by 1. On absorbing a rotational pack ($N_{rot} + 1$), $a = N_{irr}^2$ is unchanged and b increases step by step from k^2 to $k^2 + 2N + 1$; on absorbing an irreducible pack ($N_{irr} + 1$), a increases from N_{irr}^2 to $(N_{irr} + 1)^2$ and b from k^2 to $k^2 + 2N_{rot}$; what is emitted is always an irreducible capsule: a rotational capsule must first be re-registered as an irreducible capsule and is then emitted (Section 5.1); emitting one irreducible capsule decreases a by $2N_{irr} - 1$ and b by $2N_{rot}$, each sub-step decreasing $a + b$ by 1. The process state at each sub-step is called a sub-step state and is written $SS(a, b, J/\hbar)$, with $J/\hbar = 2\sqrt{ab}$ given by Eq. (24); sub-step states are not registered, and the criterion is not checked on them. The j-th sub-step of accretion $N \to N + 1$ satisfies $a + b = N^2 + j$, so the sub-step index $j = a + b - N^2$ can be read directly from a and b.

Mass and spin on sub-steps. Following Paper II, each sub-step receives energy $E_P/(2N \pm 1)$ (linear sharing). In accretion $N \to N + 1$, the mass at the j-th sub-step ($j = 1, 2, \ldots, 2N + 1$) is

$$M_j = m_P \left(N + \frac{j}{2N + 1} \right). \tag{25}$$

By Eq. (6), $\chi = Jc/(GM^2)$; substituting Eqs. (24) and (25) and using $Gm_P^2/c = \hbar$:

$$\chi_j = \frac{2\hbar\sqrt{ab}\, c}{G m_P^2 \left(N + \frac{j}{2N + 1} \right)^2} = \frac{2\sqrt{ab}}{\left(N + \frac{j}{2N + 1} \right)^2}. \tag{26}$$

For $j = 2N + 1$, $M_j = (N + 1)m_P$, in agreement with the next black-hole state. On the sub-steps $a + b = N^2 + j$, which differs from $(M_j/m_P)^2$ by $j(2N + 1 - j)/(2N + 1)^2$ and equals it only at the black-hole states at the two ends: a and b are relation counts during the process, not the entropies of an equilibrium Kerr state, and χ_j is a process quantity. In evaporation, $2N + 1$ is replaced by $2N - 1$ and j runs from high to low.

Quantization condition. By Eq. (24), the quantization condition on sub-steps is

$$\sqrt{ab} \in \mathbb{N}, \tag{27}$$

in which case $J = 2\hbar\sqrt{ab}$ is an integer multiple of $2\hbar$, and the sub-step state is called an integer sub-step state. Half-integer multiples do not occur: if $J/\hbar = m/2$, then $16ab = m^2$, so m must be a multiple of 4 and $J/\hbar$ is again even. The special case $\sqrt{ab} = 1$ (i.e. $a = b = 1$) gives the unit $2\hbar$; it lies exactly at the extremal point where the two horizons coincide ($S_+ = S_-$), does not satisfy the existence criterion, and serves only as a scale.

Absorbing a rotational pack always passes an integer sub-step state. On absorbing a rotational pack, $a = N_{irr}^2$ is a perfect square and b takes the values $k^2 + 1,\ k^2 + 2,\ \ldots,\ k^2 + 2N + 1$ in turn, so Eq. (27) holds whenever b is a perfect square. Let $m = \lfloor k \rfloor + 1$ be the smallest integer greater than k; then $m^2 > k^2$ and

$$m^2 \leq (k+1)^2 = k^2 + 2k + 1 < k^2 + 2N + 1,$$

where the last step uses $k < N$. So b necessarily passes through the perfect square m^2, and at least one integer sub-step state is passed when a rotational pack is absorbed. When a non-rotating black-hole state $BH(N, N)$ ($k = 0$) absorbs a rotational pack, the target state being $BH(N + 1, N)$, b passes through $1, 4, 9, \ldots$, and the very first sub-step is the integer sub-step state $SS(N^2, 1, J/\hbar)$, with $a = N^2$, $b = 1$ and $J/\hbar = 2\sqrt{ab} = 2N$. Here N is the sub-step $\sqrt{ab}$, which happens to equal the capsule number of the initial state; it does not mean that angular momentum is quantized in capsule numbers. For the target state, $k^2 = (N + 1)^2 - N^2 = 2N + 1$; substituting into the criterion $N_{irr} - k \geq 1$ gives $N - \sqrt{2N + 1} \geq 1$, and since both sides of $N - 1 \geq \sqrt{2N + 1}$ are positive, squaring gives the equivalent $(N - 1)^2 \geq 2N + 1$, i.e. $N(N - 4) \geq 0$, so $N \geq 4$. Hence $BH(N, N) \to BH(N + 1, N)$ is allowed for every $N \geq 4$, and every even multiple $8\hbar$, $10\hbar$, $12\hbar$, and so on, is reachable. On the other hand, along the allowed absorption paths $N_{irr} \geq 4$, so $a \geq 16$; a nonzero angular momentum requires $b \geq 1$, so $ab \geq 16$ and $J = 2\hbar\sqrt{ab} \geq 8\hbar$. Hence $8\hbar$ is the smallest nonzero value, and the angular-momentum spectrum of integer sub-step states is exactly $8\hbar$, $10\hbar$, $12\hbar$, and so on.

The smallest reachable rotating value is given by $BH(4, 4)$ absorbing a rotational pack: the first sub-step is $SS(16, 1, 8)$, with $J = 8\hbar$. This process passes through the three integer sub-step states $SS(16, 1, 8)$, $SS(16, 4, 16)$ and $SS(16, 9, 24)$, the last of which is the black-hole state $BH(5, 4)$; at the whole-pack level only the $24\hbar$ of $BH(5, 4)$ is seen (Table 1).

Order when absorbing an irreducible pack. On absorbing an irreducible pack, a increases by $2N_{irr}+1$ and b by $2N_{rot}$; the order in which the two are interleaved among the $2N + 1$ sub-steps is not fixed by the counting. This paper adopts the order that gives priority to integer sub-step quantization: among all interleavings, the one passing through the largest number of integer sub-step states.

Counting. The counting is done in black-hole states. Table 2 lists the initial states with $N = 4$–8 and their final states after absorbing a rotational pack or an irreducible pack; a final state that does not satisfy the criterion, Eq. (12), is a one-step breaking. At $N = 8$, for the first time, two initial states both give stable final states on absorbing a rotational pack.

Table 2 Initial states and their final states after absorbing a rotational pack ($N_{rot}+1$) or an irreducible pack ($N_{irr}+1$), $N = 4$–8

N	Initial states	Final states, $N_{rot}+1$	Final states, $N_{irr}+1$
4	$BH(4,4)$	$BH(5,4)$	$BH(5,5)$ (non-rotating)
5	$BH(5,5)$; $BH(5,4)$	$BH(6,5)$; $BH(6,4)$ (breaks)	$BH(6,6)$ (non-rotating); $BH(6,5)$
6	$BH(6,6)$; $BH(6,5)$	$BH(7,6)$; $BH(7,5)$ (breaks)	$BH(7,7)$ (non-rotating); $BH(7,6)$
7	$BH(7,7)$; $BH(7,6)$	$BH(8,7)$; $BH(8,6)$ (breaks)	$BH(8,8)$ (non-rotating); $BH(8,7)$
8	$BH(8,8)$; $BH(8,7)$	$BH(9,8)$; $BH(9,7)$	$BH(9,9)$ (non-rotating); $BH(9,8)$

Take all initial states with $N = 4$–40, 240 in total: 37 non-rotating states ($BH(N,N)$, one for each N) and 203 rotating states (summing Eq. (17) over $N = 4$–40).

After absorbing a rotational pack, these 240 initial states give 214 stable final states in total, and 26 break; these 26 final states do not satisfy the criterion, a one-step breaking, and come from the states of each N whose separation is closest to the lower limit, such as $BH(5,4)$, $BH(6,5)$, $BH(7,6)$ and $BH(9,7)$. The 214 initial states giving stable final states all pass at least one integer sub-step state during absorption. Of these, each of the 37 non-rotating initial states passes $\lfloor\sqrt{2N+1}\rfloor$ integer sub-step states, 223 in total, on average 6.03 per non-rotating initial state; the 177 rotating initial states pass 366 integer sub-step states in total, on average 2.07 per rotating initial state.

After absorbing an irreducible pack, these 240 initial states give 203 stable rotating final states and 37 non-rotating final states; a non-rotating initial state remains non-rotating after absorbing an irreducible pack and passes no integer sub-step state. When the 203 rotating initial states absorb an irreducible pack, a increases $2N_{irr}+1$ times and b increases $2N_{rot}$ times, and the order of interleaving decides which sub-step states are passed. With the integer-first order, 262 integer sub-step states are passed in total, on average 1.29 per rotating initial state; 149 rotating initial states pass at least one integer sub-step state, and 54 pass none.

Extending the initial states to $N \leq 60$, the greatest common divisor of $J/\hbar$ over all reachable integer sub-step states is 2, the smallest value is 8, and all even values from 8 to 120 occur; the greatest common divisor of the differences in $J/\hbar$ between successive integer sub-step states within the same absorption is also 2. The angular momentum on sub-steps is therefore quantized in units of $2\hbar$. This agrees with the proof above. Table 3 lists the integer sub-step states in some absorption processes.

Note: the black-hole state column gives the initial and final states; the number of sub-steps is $2N+1$; all final states in the table satisfy the criterion, Eq. (12). The integer sub-step states with J a multiple of $24\hbar$, $SS(16,9,24)$, $SS(64,36,96)$ and $SS(144,25,120)$, are the black-hole states $BH(5,4)$, $BH(10,8)$ and $BH(13,12)$ respectively.

Internal counting and external exchange. Two routes thus follow:

(1) Internally, energy is counted in whole packs, N and N_{irr} are integers, and the criterion is checked when a black-hole state is registered; the $J = 2\hbar N_{irr}k$ of a

Table 3 Integer sub-step states in some absorption processes (Eq. (24))

Type	Black-hole state $BH(N, N_{irr})$	Sub-steps	Integer sub-step states $SS(a, b, J/\hbar)$
$N_{rot}+1$	$(4,4) \to (5,4)$	9	(16,1,8), (16,4,16), (16,9,24)
$N_{rot}+1$	$(5,5) \to (6,5)$	11	(25,1,10), (25,4,20), (25,9,30)
$N_{rot}+1$	$(8,7) \to (9,7)$	17	(49,16,56), (49,25,70)
$N_{rot}+1$	$(9,8) \to (10,8)$	19	(64,25,80), (64,36,96)
$N_{irr}+1$	$(5,4) \to (6,5)$	11	(25,9,30)
$N_{irr}+1$	$(6,5) \to (7,6)$	13	(27,12,36)
$N_{irr}+1$	$(7,6) \to (8,7)$	15	none
$N_{irr}+1$	$(8,7) \to (9,8)$	17	(60,15,60), (64,16,64)
$N_{irr}+1$	$(9,8) \to (10,9)$	19	(68,17,68), (72,18,72), (76,19,76)
$N_{irr}+1$	$(9,7) \to (10,8)$	19	(50,32,80), (64,36,96)
$N_{irr}+1$	$(12,11) \to (13,12)$	25	(121,25,110), (144,25,120)

black-hole state is in general not an integer multiple of $\hbar$, which is the spectrum of Table 1.

(2) Externally, only the sub-steps are seen. Angular momentum is exchanged with the exterior at integer sub-step states, and the difference between two integer sub-step states,

$$\Delta J = 2\hbar \left(\sqrt{a_2 b_2} - \sqrt{a_1 b_1} \right), \tag{28}$$

is necessarily an integer multiple of $2\hbar$: both ends satisfy Eq. (27), so $\sqrt{a_1 b_1}$ and $\sqrt{a_2 b_2}$ are both integers, and so is their difference. The differences between non-integer sub-step states or the registered values of black-hole states and the integer external exchange are held temporarily in the field outside the horizon; if an absorption passes no integer sub-step state, the exchange is deferred to the next integer sub-step state.

3.7 Brief comparison with the integer quantization rule at the capsule level

Quantization itself does not require values to be integer multiples of some smallest unit: the energy levels of the hydrogen atom, $E_n \propto 1/n^2$, and the magnitude of angular momentum, $\sqrt{l(l+1)}\,\hbar$, are quantities derived from quantum numbers and are not integer multiples. The J of this framework is likewise derived from the quantization of energy (Eq. (22)). On the other hand, the standard rule requires the component of angular momentum along the spin axis to be $J_z = m\hbar$, with m an integer or half-integer. This is not a constraint of the present framework; we give only a brief comparison here.

The standard rule is the reference for the comparison. The J of the Kerr solution is itself the angular momentum along the spin axis (the conserved quantity associated with the rotational symmetry about the axis), i.e. $J = J_z$. The J of this framework is a classical macroscopic angular momentum; for large quantum numbers the ratio of the magnitude of the angular momentum to its maximal axial component tends to 1, so the comparison is meaningful in the macroscopic limit. Dividing both sides by $\hbar$

and using Eq. (6):

$$\mathcal{J} = \frac{J}{\hbar} = \frac{J_z}{\hbar} = m.$$

So the standard rule requires the numbers in the $\mathcal{J}$ column of Table 1 to be integers or half-integers.

In the variables of this framework, this requirement becomes a requirement on k. From Eq. (20), $m = 2N_{irr}k$, so $k = m/(2N_{irr})$ is a ratio of two integers; from Eq. (5), $k^2 = N^2 - N_{irr}^2$ is a positive integer, so k can only be an integer. Therefore the rotating states that satisfy the standard rule are exactly those for which (N, N_{irr}, k) is a Pythagorean triple; then m can only be an integer, and half-integers do not occur. In this framework k is in general not an integer; for example, $BH(6,5)$ has $k = \sqrt{11}$.

Two consequences follow. First, when k is an integer and $N_{irr} > k$, $N_{irr} - k \geq 1$, so criterion Eq. (12) is satisfied automatically. Second, the product of the two legs of a Pythagorean triple is divisible by 12, so $m = 2N_{irr}k$ is an integer multiple of 24, the minimum 24 being given by $BH(5,4)$.

In Table 1 only three states have integer $\mathcal{J}$: $BH(5,4)$, $BH(10,8)$ and $BH(13,12)$; the entries of Table 4 for $N \leq 13$ can all be read from Table 1.

Table 4 The spectrum of this framework and the spectrum under the standard rule

Item	This framework $\mathcal{J} = 2N_{irr}k$ (Eq. (20))	Standard rule $\mathcal{J} = m$
Total $n_{rot}(N)$, $N \leq 13$	16 (Table 1)	3: (5,4), (10,8), (13,12)
Total $n_{rot}(N)$, $N \leq 40$	203	16
Smallest prokaryotic state	(8,7), $\chi = 0.847$	(13,12), $\chi = 0.710$
Values of $\mathcal{J}$	discrete, in general non-integer	integer multiples of 24
At large N	quasi-continuous	does not become continuous

The states in both columns satisfy criterion Eq. (7). The counts in the table were obtained by enumeration: for each N, all allowed rotating states are listed from Eq. (15); the left column counts all of them, which can be checked by summing Eq. (17), $\sum_{N=5}^{40} n_{rot}(N) = 203$; the right column counts only those with integer k. The 16 states with $N \leq 40$, as $BH(N, N_{irr}, N_{rot}, k)$, are $BH(5,4,1,3)$, $BH(10,8,2,6)$, $BH(13,12,1,5)$, $BH(15,12,3,9)$, $BH(17,15,2,8)$, $BH(20,16,4,12)$, $BH(25,20,5,15)$, $BH(25,24,1,7)$, $BH(26,24,2,10)$, $BH(29,21,8,20)$, $BH(30,24,6,18)$, $BH(34,30,4,16)$, $BH(35,28,7,21)$, $BH(37,35,2,12)$, $BH(39,36,3,15)$ and $BH(40,32,8,24)$.

With the integer rule imposed at the capsule level, rotating states become much sparser, and their proportion decreases as N grows. What is compared here is the spectrum at the whole-pack level, which is internal counting. What is observable externally are the sub-steps (Section 3.6): the value $J = 2\hbar\sqrt{ab}$ at integer sub-step states satisfies the standard rule and takes only the values with even m:

$$\frac{J}{\hbar} = 2\sqrt{ab} \in \{8,\ 10,\ 12,\ \dots\}. \tag{29}$$

The sub-steps give a quantization unit of $2\hbar$, twelve times finer than the $24\hbar$ of the whole-pack states satisfying the standard rule, and compatible with the standard rule; the non-integer spectrum at the whole-pack level is internal counting and is not compared directly with external quanta.

3.8 Highest and lowest rotating spins

The lowest rotating spin takes $N_{rot} = 1$, $N_{irr} = N - 1$; then $k = \sqrt{N^2 - N_{irr}^2} = \sqrt{N^2 - (N-1)^2} = \sqrt{2N-1}$, and from Eq. (20):

$$\chi_{low}(N) = \frac{\mathcal{J}}{N^2} = \frac{2N_{irr}k}{N^2} = \frac{2(N-1)\sqrt{2N-1}}{N^2}. \tag{30}$$

It decreases monotonically with N. At large N, $N - 1 \approx N$ and $\sqrt{2N-1} \approx \sqrt{2N}$, so

$$\chi_{low}(N) \approx \frac{2N\sqrt{2N}}{N^2} = 2\sqrt{\frac{2}{N}}. \tag{31}$$

Equation (31) is only asymptotic and deviates strongly at small N: for $N = 5$ the exact value is 0.960 and the asymptotic value 1.265; for $N = 12$ they are 0.733 and 0.816.

The highest rotating spin takes the smallest N_{irr} allowed by Eq. (15); it varies with N in a sawtooth whose envelope is $\chi_{max} = 1 - 1/N^2$. At large N the allowed states are distributed quasi-continuously between χ_{low} and χ_{max}.

4 Small-N black holes: structure, growth and evaporation

4.1 Constraints on placement

A small-N black hole contains only a few E_c capsules, and how the capsules are placed inside the inner interface $R_i = N\ell_p$ decides whether it can exist as a stable structure. The constraints are as follows:

1. Capsules can exist stably only inside R_i (Papers I and II [9, 10]).
2. Capsules can touch without merging (Paper I [9]): their outer interfaces do not interpenetrate.
3. Deformation principle: a capsule as a whole deforms at constant volume, the core volume $V_i = \frac{4}{3}\pi\ell_p^3$ and the capsule volume $V_o = \frac{4}{3}\pi(2\ell_p)^3 = \frac{32}{3}\pi\ell_p^3$ each remaining unchanged; the shape is not restricted to an ellipsoid. By the isoperimetric inequality, a sphere has the smallest surface area for a given volume. Denoting by A_i, A_o the areas of the inner and outer interfaces in the spherical state and by A_i', A_o' those after deformation,

$$A_i' \geq A_i = 4\pi\ell_p^2, \qquad A_o' \geq A_o = 4\pi(2\ell_p)^2, \tag{32}$$

both interface areas are no smaller than their spherical values, with equality only for the sphere. The inner interface is the basic structure that confines the capsule, not an information-registration surface, so information registration is not impaired.

4.2 Placement of capsules

When N spherical capsules are packed into a ball of radius R_i, their centres must lie within a ball of radius $(N-2)\ell_p$, and any two centres must be at least $4\ell_p$ apart. After deformation the capsules depart from spherical shape and the shell thickness cannot be kept at ℓ_p everywhere, so these placements are all metastable. Under volume-preserving deformation the number of capsules that fit inside R_i is $\lfloor V(R_i)/V_o \rfloor = \lfloor N^3/8 \rfloor$: for $N = 2$ only one fits, and for $N \geq 3$ all N capsules fit. The placements for $N = 1$–5 are shown in Fig. 1.

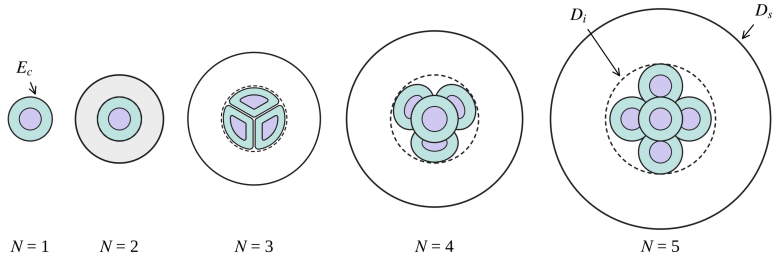

Fig. 1 Schematic placement of capsules inside black holes with $N = 1$–5 (drawn to the same scale). The large solid circle is the horizon, of diameter $D_s = 4N\ell_p$; the dashed circle is the inner interface, of diameter $D_i = 2N\ell_p$; a capsule has outer diameter $4\ell_p$ and inner diameter $2\ell_p$. $N = 3$–5 are projections of three-dimensional arrangements; apparent overlaps do not intersect in three dimensions. $N = 2$–4 are metastable states and $N = 5$ is the first stable state. For $N = 2$, the light grey between the inner interface and the horizon represents one share of energy E_P that has not formed a pack; for $N = 2$ the inner interface coincides with the outer edge of the capsule.

$N = 2$**: only one spherical capsule fits inside.** The ball of radius $R_i = 2\ell_p$ has volume $\frac{4}{3}\pi(2\ell_p)^3 = \frac{32}{3}\pi\ell_p^3$, exactly the volume of one capsule. One capsule fills R_i and can only be spherical; the second capsule has nowhere to go, and its energy E_P, not formed into a pack, is distributed between R_i and the horizon.

$N = 3$**: three deformed capsules fit inside.** If the three capsules stayed spherical, their centres would have to lie within a ball of radius $R_i - 2\ell_p = \ell_p$, so no two centres could be more than $2\ell_p < 4\ell_p$ apart; they do not fit, and must deform. Three half-planes through a common diameter, 120° apart, divide the ball of radius $R_i = 3\ell_p$ into three 120° wedges, each of volume $12\pi\ell_p^3$, larger than the capsule volume $32\pi\ell_p^3/3$; each capsule deforms and is placed in one wedge, shown in projection along the threefold axis.

$N = 4$**: four deformed capsules fit inside after deformation.** Dividing the ball of radius $R_i = 4\ell_p$ into four parts with regular-tetrahedral symmetry, one deformed capsule can be placed in each part, so all four deformed capsules fit; the figure shows the projection along a threefold axis. If the four capsules all remained

spherical, their centres would have to lie within a radius $2\ell_p$, whereas a minimum centre separation of $4\ell_p$ requires a circumradius of at least that of the regular tetrahedron, $\sqrt{6}\,\ell_p \approx 2.45\ell_p$; so spherical capsules do not fit, and they must deform.

$N \geq 5$: all N spherical capsules fit. The arrangement for $N = 5$ is given below. For $N \geq 6$, place the N centres equally spaced on a circle of radius $(N-2)\ell_p$; adjacent centres are $d_N = 2(N-2)\sin(\pi/N)\,\ell_p$ apart. For $N = 6$, $d_6 = 2{\cdot}4{\cdot}\sin(\pi/6)\,\ell_p = 4\ell_p$, so adjacent capsules just touch; and $d_N = 2\pi(1-2/N)\,[\sin(\pi/N)/(\pi/N)]\,\ell_p$. Here $1-2/N$ increases with N; $\sin x/x$ decreases monotonically on $0 < x \leq \pi/2$, and $x = \pi/N$ decreases as N grows, so $\sin(\pi/N)/(\pi/N)$ also increases with N (tending to 1). Both factors increase with N, so d_N increases with N, and for $N \geq 6$, $d_N \geq d_6 = 4\ell_p$: all N spherical capsules fit. For $N = 5$, $d_5 \approx 3.53\ell_p < 4\ell_p$ and the circle does not suffice; the triangular bipyramid below is needed. $N = 5$ is the smallest black hole in which all capsules fit inside R_i in spherical form. The five spherical capsules are arranged as a triangular bipyramid, shown in side projection. Two centres lie at the poles and three on the equator, 120° apart, all on a sphere of radius $r = 2\sqrt{2}\,\ell_p \approx 2.83\ell_p$: a pole and an equatorial centre are $\sqrt{2}\,r = 4\ell_p$ apart, and equatorial centres are $\sqrt{3}\,r \approx 4.90\ell_p$ apart, none less than $4\ell_p$; since $r < 3\ell_p$, all five spherical capsules lie inside R_i.

4.3 Stable and metastable states

Only a structure in which all capsules exist in spherical form inside R_i is a stable state. Stable and metastable here are structural definitions within the model; no dynamical stability analysis is made. There are two causes of metastability, and a structure is metastable if either applies:

1. **The shell thickness is not ℓ_p.** The separation ℓ_p between the inner and outer interfaces of E_c is the equilibrium value obtained in Paper I [9] from force balance of a spherical capsule; a sphere has the same curvature everywhere, and the thickness is ℓ_p everywhere. After deformation the curvature differs from place to place, and the thickness given by local force balance is no longer ℓ_p.
2. **One pack is in a transitional state between packing and unpacking.** The last share of energy E_P fails to form a pack inside R_i and is distributed between R_i and the horizon. It has already been registered on the horizon and is counted in N; on reaching R_i it packs and unpacks again, remaining in transition between the two.

The three metastable states have two kinds of cause: $N = 2$ is metastable only because it lies exactly in the transition between packing and unpacking, and is the smallest transitional state at the black-hole level; $N = 3$ and $N = 4$ only because of deformation. At $N = 5$ both causes of metastability disappear together. Hence $N = 5$ is both the first stable complete structure and the minimal rotating state $BH(5, 4)$: structural completeness and the onset of rotation occur at the same N. These two facts come respectively from internal packing and from the existence criterion (Eq. (16)), and are two independent results.

4.4 Step-by-step growth and Ostwald's rule of stages

Given sufficient supply and a suitable temperature, a small black hole can grow step by step from one metastable state to the next:

$$2\,(1+1) \;\to\; 3\,(\text{deformed}) \;\to\; 4\,(\text{deformed}) \;\to\; 5\,(\text{stable, can rotate}) \;\to\; \cdots \tag{33}$$

There are two conditions. Supply: the energy needed to form packs must be delivered continuously, which corresponds to a supersaturated state and is available only near the contact density of the early universe. Temperature: the background temperature must exceed the packing temperature $T_N^+ = E_P/[4\pi(2N+1)k_B]$ (Eq. (2)) for net absorption. If growth is slower than the rise of the critical mass $M_{crit}(t)$, the black hole turns to net evaporation and steps back last in, first out, $3 \to 2 \to 1$.

This process is structurally analogous to Ostwald's rule of stages: in phase transitions a metastable phase often appears first and then transforms step by step into the stable phase [12, 13]. The mechanisms differ: in phase transitions the metastable phase appears first because its nucleation barrier is lower; here it is because the last pack does not fit inside R_i, or fits only after deformation, which is a geometric cause.

Step-by-step growth and direct formation at the mean density ρ_c/N^2 (Section 2) are two coexisting routes, corresponding respectively to a stepwise phase transition and to direct nucleation.

4.5 Breaking and ordered growth in accretion

Step-by-step growth requires the black hole to remain in an allowed state after absorbing each pack, which a rotating small black hole cannot always achieve. Each capsule entering or leaving is registered as a whole in either the irreducible part or the rotational part, so there are only two kinds of accretion: rotational packs and irreducible packs (two kinds of registration, not two kinds of capsule; see Section 5.1).

One-step breaking. From Eq. (10), the separation of the two horizons is $g = R_{A+} - R_{A-}$. Substituting $R_{A+} = 2N_{irr}\ell_p$ and $R_{A-} = 2\ell_p\sqrt{N^2 - N_{irr}^2}$ from Eq. (4):

$$g = 2\ell_p\left(N_{irr} - \sqrt{N^2 - N_{irr}^2}\right) = 2\ell_p\,(N_{irr} - k)\,, \tag{34}$$

where k is defined by Eq. (5) and $R_{A-} = 2\ell_p k$.

Accreting one rotational pack: the separation shrinks. $N \to N+1$ with N_{irr} unchanged. Substituting $N+1$ into Eq. (34) gives the new separation

$$g' = 2\ell_p\left(N_{irr} - \sqrt{(N+1)^2 - N_{irr}^2}\right). \tag{35}$$

Since $(N+1)^2 - N_{irr}^2 > N^2 - N_{irr}^2$, $g' < g$: a rotational pack always shrinks the separation. Checking the rotating states with $N \le 8$ one by one (old separation from Eq. (34), new separation from Eq. (35)), with the results listed in Table 5:

The target states below $2\ell_p$ in the table are formal target states: they do not satisfy the criterion, breaking occurs, and so no stable state can form; for the initial

Table 5 Separation after a rotating state with $N = 5$–8 absorbs one rotational pack

Initial state	Old g/ℓ_p	After +1 rotational pack	New g'/ℓ_p	Result
(5,4)	2.00	(6,4)	−0.94	below $2\ell_p$; passes the extremal state; breaks
(6,5)	3.37	(7,5)	0.20	below $2\ell_p$; breaks
(7,6)	4.79	(8,6)	1.42	below $2\ell_p$; breaks
(8,7)	6.25	(9,7)	2.69	still allowed; one more rotational pack, to (10,7), passes the extremal state

state $BH(5,4)$, the formal target state $BH(6,4)$ after one rotational pack even has a negative separation ($R_{A-} > R_{A+}$). So the single rotating state of each of $N = 5$, 6 and 7 breaks on absorbing one more rotational pack; $N = 7$ is the last N with one-step breaking.

Accreting one irreducible pack: the separation grows. For an irreducible pack, $N \to N+1$ and $N_{irr} \to N_{irr}+1$. Substituting into Eq. (34), the quantity under the square root becomes

$$(N+1)^2 - (N_{irr}+1)^2 = N^2 - N_{irr}^2 + 2\,(N - N_{irr}) = k^2 + 2N_{rot},$$

so

$$g' = 2\ell_p \left(N_{irr} + 1 - \sqrt{k^2 + 2N_{rot}}\right), \qquad g' - g = 2\ell_p \left[1 - \left(\sqrt{k^2 + 2N_{rot}} - k\right)\right]. \tag{36}$$

The condition for $g' \geq g$ is

$$\sqrt{k^2 + 2N_{rot}} \leq k + 1 \quad \Leftrightarrow \quad N_{rot} \leq k + \frac{1}{2}. \tag{37}$$

From Eqs. (3) and (5), $k^2 = N_{rot}^2 + 2N_{irr}N_{rot} \geq N_{rot}^2$, hence $N_{rot} \leq k$, and Eq. (37) always holds. So irreducible packs are always safe: a rotating state remains allowed after absorbing an irreducible pack, e.g. $BH(5,4) \to BH(6,5) \to BH(7,6) \to BH(8,7)$; only rotational packs push a black hole toward breaking.

Ordered accretion. For a small black hole to grow, accretion must be ordered: rotating states with $N \leq 7$ can absorb only capsules registered in the irreducible part. The registration type is determined by the angular momentum each capsule brings in (Section 5.4), and the black hole cannot choose: only when every infalling capsule brings in no more angular momentum than the irreducible registration ΔJ_{irr} are they all registered in the irreducible part, and only then can the rotating small black hole grow; as soon as one capsule is registered in the rotational part, breaking occurs. A non-rotating black hole remains non-rotating on absorbing an irreducible pack, and enters $BH(N+1, N)$ ($N \geq 4$) on absorbing a rotational pack. This is a further reason why small black holes find it hard to grow: besides the two conditions of supply and temperature in Section 4.4, accretion must be ordered.

4.6 Evaporation: lifetime and early loss of angular momentum

Evaporation lifetime. The commonly used evaporation lifetime of a Schwarzschild black hole, $t_{ev,\gamma} = 5120\pi G^2 M^3/(\hbar c^4)$, counts only photons and assumes blackbody radiation from the horizon surface; it applies only to large black holes with very low Hawking temperature. The temperature at which a small-N black hole emits a whole pack is given by Eq. (2), $k_B T_N^- = E_P/[4\pi(2N-1)]$; this is the characteristic energy of the Hawking radiation, not the total energy NE_P of the black hole. Massless particles (photons, gluons, gravitons) are emitted at any temperature; massive particles are emitted copiously only when their rest energy is below about $k_B T_N^-$ [14, 15]. For the last pack ($N = 2 \to 1$), $k_B T_2^- = E_P/(12\pi) \approx 3.2 \times 10^{17}$ GeV ($T_2^- \approx 3.8 \times 10^{30}$ K), far above the top quark (about 173 GeV), which has the highest rest energy among the elementary particles of the Standard Model; so all Standard Model particles are emitted, and all at ultra-relativistic speeds:

$$m_{max}c^2 \approx k_B T_2^- \approx 3.2 \times 10^{17}\ \text{GeV}. \tag{38}$$

The curved spacetime outside a black hole reflects part of the low-energy quanta back, so both absorption and emission are less than for a blackbody; this is called a greybody. On the basis of the greybody calculation of MacGibbon and Webber [14] for all Standard Model particles, Carr et al. [16, 17] give the accurate evaporation lifetime

$$t_{ev,grey} \approx 6.25 \times 10^{-18}\,\frac{M^3}{f_{ev}}, \tag{39}$$

where M is in kg and $t_{ev,grey}$ in s; f_{ev} is the weight of the emitted particle species: $f_{ev} = 1$ when only massless particles (photons, three generations of neutrinos, gravitons) are emitted, and $f_{ev} = 15.35$ when all Standard Model particles are included. For small-N black holes we take $f_{ev} = 15.35$ and substitute $M = Nm_P$ ($m_P = 2.176 \times 10^{-8}$ kg):

$$t_{ev,grey} \approx \frac{6.25 \times 10^{-18}}{15.35} \times \left(2.176 \times 10^{-8}\right)^3 N^3\ \text{s} = 4.19 \times 10^{-42} N^3\ \text{s} = 77.8\, N^3 t_P. \tag{40}$$

With $t_P = 5.391 \times 10^{-44}$ s, $t_{ev,grey} \approx 3.4 \times 10^{-41}$ s for $N = 2$, $t_{ev,grey} \approx 5.2 \times 10^{-40}$ s for $N = 5$, and $t_{ev,grey} \approx 1.4 \times 10^{-39}$ s for $N = 7$. Counting only photons, the coefficient is $5120\pi \approx 16085$ and the lifetime is about 207 times longer. For $N = 2$–7 this is an estimate obtained by extrapolating the semiclassical formula to the Planck scale. $f_{ev} = 15.35$ counts particles only up to about 10^3 GeV; if many more species exist at higher energies, the lifetime is shorter still.

Growth and evaporation of black holes in the early universe. Small-N black holes form in the early universe and absorb on net while the background temperature T_{bg} is above T_H. Figure 2 integrates the net change of black-hole mass: absorption of background radiation and emission of Hawking radiation use the same cross-section, so the net change vanishes exactly at $T_H = T_{bg}$, i.e. at the critical mass $M_{crit}(t)$; the cross-section is taken as 0.574 times the geometric-optics value so that

the emission-only lifetime agrees with Eq. (40): with emission only, the black-body lifetime is $40960\pi N^3 t_P/(27 s g_*)$; setting it equal to the $77.8N^3 t_P$ of Eq. (40) with $g_* = 106.75$ gives $s = 40960\pi/(27 \times 106.75 \times 77.8) \approx 0.574$. Black holes with $N_0 = 2$–6 all form at $t_0 = 20\,t_P$, and growth by accretion is capped at three times the initial mass (not reached in this figure). Each black hole first grows and, once overtaken by $M_{crit}(t)$, turns to net evaporation. For initial $N_0 = 2$, 3, 4, 5, 6, N grows at most to about 2.2, 3.7, 5.5, 7.7 and 10.5 respectively, and the black holes evaporate completely at about 4.8×10^{-41}, 2.3×10^{-40}, 7.4×10^{-40}, 2.0×10^{-39} and 5.0×10^{-39} s. In this framework energy enters and leaves in whole packs of E_P, and the continuous curves represent average behaviour. Counting in whole packs, $N_0 = 2$ and 3 absorb less than one full pack, cannot grow to the next level, and evaporate directly last in, first out; $N_0 = 4$ can grow to 5, the first stable structure; $N_0 = 5$ and 6 can grow to 7 and 10 respectively. This does not contradict Eq. (33): the step-by-step growth of Eq. (33) requires a pack-forming supply near the contact density (Section 4.4), whereas Fig. 2 counts only absorption of background radiation; with background radiation alone, $N = 2$ and 3 cannot grow along Eq. (33).

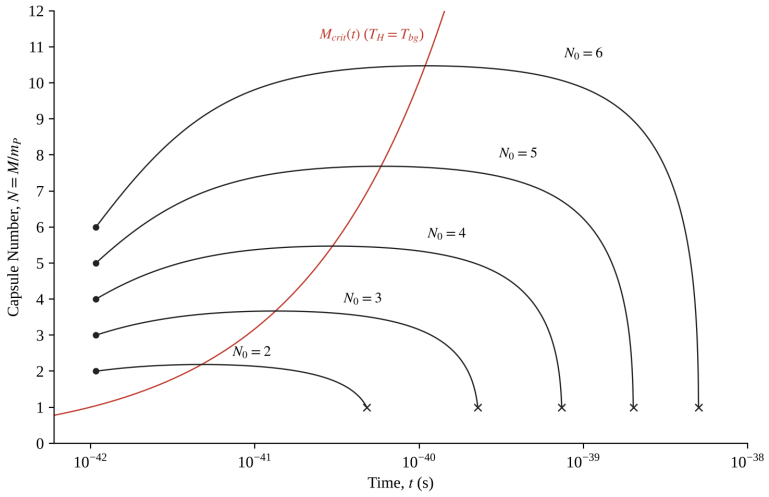

Fig. 2 Growth and evaporation of small-N black holes in the early universe. The red line is the critical mass $M_{crit}(t)$ $(T_H = T_{bg})$; dots mark formation ($t_0 = 20\,t_P$, $N_0 = 2$–6) and crosses complete evaporation. Emission and absorption use the same cross-section, normalized to the all-particle grey-body lifetime (Eq. (40)).

Loss of angular momentum in evaporation. When a rotating small-N black hole evaporates, what is emitted is always an irreducible capsule; a rotational capsule must first be re-registered as an irreducible capsule and is then emitted (Section 5.1). The two rules operate in parallel: re-registration before emission prescribes how a capsule is emitted, and the existence criterion decides whether the black-hole state after each registration is allowed; re-registration increases the separation, so the intermediate state always satisfies the criterion, and the criterion forbids the direct emission of

an irreducible capsule only when the separation is close to the lower limit, in which case the state must first be re-registered. There are two possibilities: a rotational capsule is first re-registered and then emitted, $BH(N, N_{irr}) \to BH(N-1, N_{irr})$, with intermediate state $BH(N, N_{irr}+1)$ and the same final state as emitting a rotational pack; the separation increases and this is always allowed; or an irreducible pack is emitted directly, $BH(N, N_{irr}) \to BH(N-1, N_{irr}-1)$, which decreases the separation. A state whose separation is close to the lower limit falls into the forbidden region if it emits directly: $BH(5,4) \to BH(4,3)$, and $N \leq 4$ cannot rotate; $BH(9,7) \to BH(8,6)$, $g = 1.42\ell_p$; $BH(12,9) \to BH(11,8)$, $g = 0.90\ell_p$ (all below $2\ell_p$). So the first step of these states must be a re-registration. The other states may also emit directly, as long as every step still satisfies the criterion; for example, $BH(7,6) \to BH(6,5) \to BH(5,4)$ emits directly twice. And since $N \leq 4$ cannot rotate, a rotating small black hole must shed all of its angular momentum before evaporating down to $N = 4$.

Take $BH(5,4)$ as an example. $BH(5,4)$ is re-registered as $BH(5,5)$: N is unchanged, and the $2N_{irr}+1 = 9$ relations of the inner horizon ($S_- = 4\pi k_B(N^2 - N_{irr}^2) = 36\pi k_B$) are all transferred to the outer horizon, S_+ rising from $64\pi k_B$ to $100\pi k_B$; by Eq. (22), J falls from $24\hbar$ to 0, the difference being held first by the field. By last in, first out, the first capsule emitted is exactly this re-registered one, $BH(5,5) \to BH(4,4)$: $2N-1 = 9$ sub-steps each emit entropy $4\pi k_B$ from the outer horizon, S_+ returns to $64\pi k_B$, M falls from $5m_P$ to $4m_P$, and the angular momentum held in the field is carried away by radiation. Taken together, one pack E_P carries away exactly all of the angular momentum, with no net change of the outer-horizon area. Thereafter the black hole evaporates pack by pack as a non-rotating black hole, $4 \to 3 \to 2 \to 1$.

At large N the great majority of rotating states lie far from the boundary, the criterion is satisfied with or without re-registration, and it does not prescribe the number of re-registrations (Section 5.6).

Sub-steps and emitted quanta. By Paper II [10], unpacking $N \to N-1$ proceeds in $2N-1$ sub-steps, each emitting energy $E_P/(2N-1)$ and entropy $4\pi k_B$, in about $4\pi/2.70 \approx 4.65$ thermal quanta (estimated for photons). The species emitted are decided by the Hawking temperature: every particle whose rest energy mc^2 is below about $k_B T_N^-$ is emitted (Eq. (38)). So the emission includes not only massless particles such as photons, neutrinos and gravitons, but also massive particles such as electrons, quarks, gluons, W and Z bosons and the Higgs boson. All are emitted at ultra-relativistic speeds, so their energy and entropy can be computed as for massless particles, and the effective number of degrees of freedom $g_* = 106.75$ includes all Standard Model particles. If particles beyond the Standard Model exist at this energy scale, they are emitted as well. For example, $5 \to 4$ proceeds in 9 sub-steps, emitting about $9 \times 4.65 \approx 42$ quanta. In a thermal-radiation estimate, the unpacking temperature at $N = 5$ is $k_B T_5^- = E_P/(36\pi)$, the mean energy per quantum is about $2.70 k_B T_5^- \approx 0.024 E_P$, and one pack E_P is divided into about 42 quanta (estimated for photons), consistent with the sub-step estimate.

Conclusion. Small black holes with $N \leq 7$ can grow only if accretion is ordered, and a rotating state breaks on absorbing one rotational pack. In evaporation, a rotating state whose separation is close to the lower limit must first be re-registered and then emitted; every rotating state must shed all of its angular momentum before evaporating down to $N = 4$, and thereafter evaporates pack by pack as a non-rotating black hole. In the early universe they disappear within about 10^{-38} s (Fig. 2), and the emitted energy enters the early universe. Black holes with $N \leq 4$ are even less stable because their structure is metastable (Section 4.3).

5 Gain and loss of angular momentum

5.1 Registration rule for E_c capsules entering and leaving a black hole

For each pack E_P that enters, $N_{irr} + 1$ or $N_{rot} + 1$; for each pack E_P that leaves, what is emitted is always an irreducible capsule, $N_{irr} - 1$; a rotational capsule must first be re-registered as an irreducible capsule and is then emitted ($N_{rot} - 1$, $N_{irr} + 1$, N unchanged). A re-registered capsule becomes the last pack registered on the outer horizon, and by last in, first out the last pack re-registered is the first to be emitted. The energy of a capsule is indivisible and must be registered as a whole pack. Energy determines how many times registration occurs; angular momentum determines in accretion which term it is registered in, and in emission whether re-registration is needed, according to the instantaneous requirement. Re-registration in the reverse direction ($N_{irr} \to N_{rot}$) would decrease the outer-horizon entropy at fixed N, violating the second law, and does not occur; so in accretion a rotational capsule is registered directly on the inner horizon. The outer horizon registers the irreducible part ($S_+ = 4\pi k_B N_{irr}^2$), and the inner horizon the rotational part together with the cross relations ($S_- = 4\pi k_B(N_{rot}^2 + 2N_{irr}N_{rot})$). The registration type is decided locally for each capsule where it enters. All capsules inside a black hole are identical; they have no spin of their own and only co-rotate with the black hole as a whole. "Rotational pack" and "irreducible pack" are not two kinds of capsule but two kinds of registration: when a pack enters or leaves, what changes is either the count of the rotational part ($N_{rot} \pm 1$) or that of the irreducible part ($N_{irr} \pm 1$). The difference lies in where the relations are registered: the cross relations between rotational and irreducible capsules are registered on the inner horizon.

E_c is the limit point $N = 1$ of the Schwarzschild sequence and has no spin when alone ($N \leq 4$ cannot rotate). Once inside a black hole, a capsule rotates along with it as part of the whole; angular momentum is a property of the whole. From Eq. (3), the outer horizon registers the number of relations among irreducible capsules, $N_{irr}^2 = S_+/(4\pi k_B)$, and the inner horizon the number of relations involving

rotational capsules, $k^2 = N_{rot}^2 + 2N_{irr}N_{rot} = S_-/(4\pi k_B)$; therefore

$$\begin{aligned} J &= 2\hbar N_{irr} k = 2\hbar\sqrt{N_{irr}^2\, k^2} = 2\hbar\sqrt{\frac{S_+}{4\pi k_B}\cdot\frac{S_-}{4\pi k_B}} \\ &= \frac{\hbar}{2\pi k_B}\sqrt{S_+ S_-} = \frac{c^3}{8\pi G}\sqrt{A_+ A_-}, \end{aligned} \tag{41}$$

i.e. the angular momentum is proportional to the geometric mean of the two sets of relation counts; it vanishes if either is absent, so it cannot be apportioned to individual packs. The last step uses $S_\pm = k_B A_\pm/(4\ell_p^2)$ and $\ell_p^2 = \hbar G/c^3$, where $A_\pm$ are the areas of the two horizons. This is the known area-product formula $A_+A_- = (8\pi GJ/c^3)^2$ [18, 19]: the product of the areas of the two horizons is independent of the mass and is determined by the angular momentum alone. When one pack enters or leaves, the change in total angular momentum depends on its registration type: rotational packs and irreducible packs differ (Section 5.2). This change in angular momentum, computed from the registration type, is called the registered value below; what is registered is the capsule, i.e. the relation counts on the two horizons, while the angular momentum is derived from the entropies of the two horizons (Eq. (41)) and is not itself registered. Mass-energy and angular momentum are global quantities of the black-hole interior together with its exterior field; externally they appear as frame dragging, and rotational energy can be extracted through the ergosphere outside the horizon [20, 21]. Registration is updated in whole packs, and the difference between the registered angular momentum and the angular momentum actually brought in or carried out is held temporarily by the field outside the horizon (including the gravitational field itself).

Internal counting and external observation. Registration proceeds in units of whole packs: N and N_{irr} change only when a pack is completed, and the whole-pack registered angular momentum is finalized only then; the external exchange of angular momentum takes place on the sub-steps. Seen from outside, the entry or exit of one pack is divided into $2N \pm 1$ sub-steps (+ for absorption, − for emission), each bringing in or carrying out energy $E_P/(2N \pm 1)$ and entropy $4\pi k_B$. Counted by the entropy associated with the sub-steps: on absorbing a rotational pack all $2N + 1$ sub-steps correspond to changes in the inner-horizon entropy S_-; on absorbing an irreducible pack, $2N_{irr} + 1$ correspond to changes in the outer-horizon entropy S_+ and $2N_{rot}$ to changes in S_-; on emitting an irreducible capsule, $2N_{irr} - 1$ correspond to S_+ and $2N_{rot}$ to S_-. A re-registration before emission moves $2N_{irr} + 1$ relations from the inner horizon to the outer horizon at fixed N, without sub-steps. What is observed externally is the sequence of sub-steps; on average, the ratio of angular momentum to energy per sub-step is the same as for the whole pack. The external exchange of angular momentum takes place at integer sub-step states satisfying Eq. (27), each time by an integer multiple of $2\hbar$ (Section 3.6). Accumulated over many sub-steps, this appears as a continuous dJ/dM, the level described by general relativity and semiclassical calculations; whole-pack registration is the internal counting of this continuous change.

Thermodynamic tendency and kinetic control. In a re-registration ($N_{rot} \to N_{irr}$), N is unchanged and N_{irr} increases by 1. By Eq. (3), the outer-horizon entropy

increases by $\Delta S_+ = 4\pi k_B(2N_{irr}+1)$ and the inner-horizon entropy decreases by the same amount, so the sum $S_+ + S_- = 4\pi k_B N^2$ is unchanged. What increases is the outer-horizon entropy, i.e. the entropy of the Kerr black hole; at fixed N, S_+ has the non-rotating value $4\pi k_B N^2$ as its upper limit ($N_{irr} \to N$), and re-registration moves it towards this limit. Re-registration is therefore a thermodynamic tendency, a self-deformation at fixed N that does not depend on the external environment. This tendency agrees with the general tendency of deformable rotating bodies to lower their rotational energy through deformation and dissipation; for black holes its precise form is that, in classical processes obeying the area theorem, the irreducible mass never decreases [20]: rotational energy can be extracted or degraded into irreducible mass-energy, but not the reverse. Re-registration is exactly the degradation of rotational energy into irreducible mass-energy, which is also why re-registration in the reverse direction does not occur. Re-registration proceeds in whole packs like registration, and both re-registration and emission come in units of one pack E_P: (N, N_{irr}) correspond directly to energy, N to the total mass-energy and N_{irr} to the irreducible mass-energy; re-registering one pack moves one share E_P from the rotational part to the irreducible part, and emitting one pack emits one share E_P. The angular-momentum differences caused by re-registration and emission are first held by the field outside the horizon and then carried away by radiation; before each re-registration, it is checked whether the difference held in the field has been cleared, and only then is the next pack re-registered, as in feedback control. Before emission there is no way to know in advance how many packs in total must be re-registered; re-registration can only follow the instantaneous requirement. The radiation carries angular momentum away from the black hole and is the rate-controlling step: when angular momentum flows out quickly, re-registration is frequent; when it flows out slowly, re-registration is sparse. The direction is decided by thermodynamics and the rate by kinetics; the actual evolution is under kinetic control.

5.2 Change in angular momentum caused by the two kinds of pack

Substituting Eq. (5) into Eq. (22):

$$J = 2\hbar N_{irr} k. \tag{42}$$

Exact expressions. A rotational pack added or removed changes $N \to N \pm 1$ with N_{irr} unchanged; combining Eqs. (22) and (42),

$$\begin{aligned} \Delta J^{\pm}_{rot} &= 2\hbar N_{irr}\sqrt{(N\pm 1)^2 - N_{irr}^2} - 2\hbar N_{irr} k \\ &= 2\hbar N_{irr}\left[\sqrt{(N\pm 1)^2 - N_{irr}^2} - k\right]. \end{aligned} \tag{43}$$

An irreducible pack added or removed changes $N \to N \pm 1$ and $N_{irr} \to N_{irr} \pm 1$. First compute the new k:

$$\begin{aligned}(N \pm 1)^2 - (N_{irr} \pm 1)^2 &= N^2 \pm 2N + 1 - N_{irr}^2 \mp 2N_{irr} - 1 \\ &= \left(N^2 - N_{irr}^2\right) \pm 2\left(N - N_{irr}\right) = k^2 \pm 2N_{rot},\end{aligned}$$

so

$$\Delta J_{irr}^{\pm} = 2\hbar \left[(N_{irr} \pm 1) \sqrt{k^2 \pm 2N_{rot}} - N_{irr} k \right]. \tag{44}$$

The upper signs (+ for $\pm$, − for $\mp$) correspond to absorption and the lower signs (− for $\pm$, + for $\mp$) to emission. **Absorption increases J; emission decreases J.** For Eq. (43) with the upper sign, a rotational pack has $\sqrt{(N+1)^2 - N_{irr}^2} > k$, so $\Delta J_{rot}^{+} > 0$; for Eq. (44) with the upper sign, an irreducible pack has $\sqrt{k^2 + 2N_{rot}} \geq k$ and $N_{irr} + 1 > N_{irr}$, so $\Delta J_{irr}^{+} \geq 0$, with equality only for $N_{rot} = 0$, $k = 0$ (non-rotating). For Eq. (43) with the lower sign, the quantity under the square root for a rotational pack is smaller than k^2; for Eq. (44) with the lower sign, neither factor of an irreducible pack increases; so $\Delta J_{rot}^{-} < 0$ and $\Delta J_{irr}^{-} \leq 0$. Every capsule brings in energy E_P, so accretion cannot decrease J; J can decrease by re-registration (N unchanged), by reducing the number of capsules (evaporation, the Penrose process, superradiance; Section 5.6) or by breaking.

Change in spin. From Eq. (6), $\chi = \mathcal{J}/N^2 = \sin 2\omega$; from Eq. (4), $\cos\omega = N_{irr}/N$, and for $\omega < \pi/4$ a larger ω gives a larger $\chi = \sin 2\omega$. For a rotational pack N_{irr} is unchanged: on absorption N increases, $\cos\omega$ decreases and χ increases; on emission the reverse. For an irreducible pack, on absorption $(N_{irr}+1)/(N+1) > N_{irr}/N$ (since $N_{irr} < N$), $\cos\omega$ increases and χ decreases; on emission $(N_{irr}-1)/(N-1) < N_{irr}/N$ and χ increases. The four cases are shown in Table 6.

Table 6 Changes in J and χ on absorbing and emitting one pack (exact values)

Pack	Process	Example	$J/\hbar$	χ
Rotational	absorption	$(8,7) \to (9,7)$	$54.2 \to 79.2$, increases	$0.847 \to 0.978$, increases
Rotational	emission	$(9,7) \to (8,7)$	$79.2 \to 54.2$, decreases	$0.978 \to 0.847$, decreases
Irreducible	absorption	$(8,7) \to (9,8)$	$54.2 \to 66.0$, increases	$0.847 \to 0.814$, decreases
Irreducible	emission	$(9,8) \to (8,7)$	$66.0 \to 54.2$, decreases	$0.814 \to 0.847$, increases

In the table, rotational emission denotes the net result of re-registration to $(9,8)$ followed by the emission of an irreducible pack, not the direct emission of a rotational capsule; Eq. (43) gives exactly the net change of this composite process. The signs of the changes hold exactly for all N. Emitting an irreducible pack directly decreases N and increases ω, both of which shrink the separation $g = 2N\ell_p(\cos\omega - \sin\omega)$. So a state whose separation is close to the lower limit $2\ell_p$ would fall below the limit and break if it emitted directly; it can only be re-registered first, increasing the separation and moving away from breaking, and then emit (Section 4.6). This order, re-registration before emission, is decided by the existence criterion.

5.3 Large-N approximation and the angular momentum actually brought in

Large-N approximation. At large N, Eqs. (43) and (44) are differences of two nearly equal large numbers, and their relation to the spin is not apparent. When N is very large at fixed nonzero spin (k of the same order as N, $(2N+1)/k^2 \ll 1$), adding 1 can be treated as a small change and a first-order approximation used, denoted below by $\simeq$.

Baseline. From Eq. (6), $J = \chi N^2 \hbar$. When one pack is absorbed or emitted, both N and χ change; to first order by the product rule,

$$\Delta J = \hbar\, \Delta\left(\chi N^2\right) = \hbar\left(N^2\, \Delta\chi + 2\chi N\, \Delta N\right).$$

Taking $\Delta N = \pm 1$, dividing both sides by $N\hbar\,\Delta N$, and using $\Delta \ln M = \Delta N/N$:

$$\frac{|\Delta J|}{N\hbar} = 2\chi + \frac{\Delta\chi}{\Delta \ln M}. \tag{45}$$

So $|\Delta J| = 2\chi N\hbar$ is the baseline: with registration in proportion to composition ($p = N_{rot}/N$), the mean spin is unchanged to first order; the difference between the angular-momentum change of any pack, $|\Delta J|/(N\hbar)$, and 2χ is the $\Delta\chi/\Delta \ln M$ caused by that pack, and $\Delta\chi/\Delta \ln M$ is determined only by the geometry of the horizon circle. Taking differentials of $\chi = \sin 2\omega$ and of $\cos\omega = N_{irr}/N$:

$$\Delta\chi = \Delta\left(\sin 2\omega\right) = 2\cos 2\omega\, \Delta\omega, \tag{46}$$

$$\Delta\cos\omega = -\sin\omega\, \Delta\omega = \Delta\left(N_{irr}/N\right) = \frac{\Delta N_{irr} - \cos\omega\, \Delta N}{N}. \tag{47}$$

Rotational pack ($\Delta N_{irr} = 0$): substituting into Eq. (47),

$$-\sin\omega\, \Delta\omega \simeq -\frac{\cos\omega\, \Delta N}{N} \;\Rightarrow\; \Delta\omega \simeq \frac{\cos\omega}{\sin\omega}\,\frac{\Delta N}{N} = \cot\omega\,\frac{\Delta N}{N}.$$

From Eq. (46) and $\Delta \ln M = \Delta N/N$:

$$\frac{\Delta\chi}{\Delta \ln M} = \frac{N}{\Delta N}\cdot 2\cos 2\omega\, \Delta\omega = \frac{N}{\Delta N}\cdot 2\cos 2\omega\cot\omega\,\frac{\Delta N}{N} = 2\cot\omega\cos 2\omega.$$

Substituting into Eq. (45):

$$\frac{\Delta J_{rot}}{N\hbar} \simeq 2\sin 2\omega + 2\cot\omega\cos 2\omega = \frac{2\cos\omega}{\sin\omega}\left(2\sin^2\omega + \cos^2\omega - \sin^2\omega\right) = 2\cot\omega. \tag{48}$$

Irreducible pack ($\Delta N_{irr} = \Delta N$): substituting into Eq. (47),

$$-\sin\omega\, \Delta\omega \simeq \frac{(1-\cos\omega)\, \Delta N}{N} \;\Rightarrow\; \Delta\omega \simeq -\frac{1-\cos\omega}{\sin\omega}\,\frac{\Delta N}{N} = -\tan\frac{\omega}{2}\,\frac{\Delta N}{N},$$

where the last step uses the half-angle formula $\tan(\omega/2) = (1-\cos\omega)/\sin\omega$. From Eq. (46) and $\Delta \ln M = \Delta N/N$:

$$\frac{\Delta\chi}{\Delta \ln M} = \frac{N}{\Delta N}\cdot 2\cos 2\omega\,\Delta\omega = \frac{N}{\Delta N}\cdot 2\cos 2\omega\left(-\tan\frac{\omega}{2}\right)\frac{\Delta N}{N} = -2\cos 2\omega\tan\frac{\omega}{2}.$$

Substituting into Eq. (45):

$$\frac{\Delta J_{irr}}{N\hbar} \simeq 2\sin 2\omega - 2\cos 2\omega\tan\frac{\omega}{2}. \tag{49}$$

For allowed states $\omega < \pi/4$ and $\cos 2\omega > 0$: rotational packs lie above the baseline and irreducible packs below it, in agreement with the exact conclusion of Section 5.2 (Table 6). The magnitudes in this first-order approximation apply equally to absorption and emission, but are accurate only at large N: for $N \sim 10^6$ they agree with the exact values to four significant figures; at small N the deviation for rotational packs is large. For example, when $BH(8,7)$ emits its only rotational pack and becomes $BH(7,7)$, the exact value is $\Delta J/\hbar = -54.2$ while the first-order approximation gives -28.9; Eqs. (43) and (44) must then be used.

In general relativity, matter in an accretion disk moves inward gradually along circular orbits and falls into the black hole after reaching the innermost stable circular orbit (ISCO). Denote by $x = r_{isco}c^2/(GM)$ the ratio of the ISCO radius r_{isco} to GM/c^2 (dimensionless; $x = 6$ without rotation, i.e. $r_{isco} = 3R_s$). For a prograde disk [6]

$$x = 3 + Z_2 - \sqrt{(3-Z_1)\,(3+Z_1+2Z_2)}, \tag{50}$$

where $Z_1 = 1 + (1-\chi^2)^{1/3}\left[(1+\chi)^{1/3} + (1-\chi)^{1/3}\right]$ and $Z_2 = \sqrt{3\chi^2 + Z_1^2}$.

Disk matter of mass m moving inward along circular orbits radiates away mass $(1-\tilde{e})\,m$ on the way, and its angular momentum is also transported outward by torques within the disk, part of it carried off by radiation; on reaching the ISCO the remaining mass is $\tilde{e}\,m$ and the angular momentum $\tilde{l}\,m$. After reaching the ISCO all the matter falls through the horizon, so the black hole gains mass $\tilde{e}\,m$ and angular momentum $\tilde{l}\,m$. We call $\tilde{e}$ the accretion coefficient and $\tilde{l}$ the specific angular momentum per unit mass of disk matter m [6]; the radiative efficiency of accretion is $\eta = 1-\tilde{e}$:

$$\tilde{e} = \sqrt{1-\frac{2}{3x}}, \qquad \tilde{l} = \frac{2}{3\sqrt{3}}\frac{GM}{c}\left(1+2\sqrt{3x-2}\right). \tag{51}$$

Each capsule absorbed increases the black-hole mass by m_P. From $\tilde{e}\,m = m_P$, the disk mass that must fall in is $m = m_P/\tilde{e}$, and the angular momentum brought in is $\tilde{l}\,m = \tilde{l}\,m_P/\tilde{e}$; multiplying numerator and denominator by GM/c makes the coefficient dimensionless:

$$L_{in} = \tilde{l}\cdot\frac{m_P}{\tilde{e}} = \frac{\tilde{l}c}{\tilde{e}GM}\cdot\frac{GMm_P}{c}. \tag{52}$$

Moreover $GMm_P/c = GNm_P^2/c = N\hbar$ (using $m_P^2 = \hbar c/G$), so

$$L_{in} = \frac{\tilde{l}c}{\tilde{e}GM}\,N\hbar. \tag{53}$$

The two ends: without rotation, $\chi \to 0$, $x = 6$, $\tilde{e} = 2\sqrt{2}/3 \approx 0.943$ ($\eta \approx 5.7\%$), $\tilde{l} = 2\sqrt{3}\,GM/c$, and $L_{in} = 3\sqrt{3/2}\,N\hbar \approx 3.67N\hbar$; for $\chi \to 1$, $x \to 1$, $\tilde{e} \to 1/\sqrt{3} \approx 0.577$ ($\eta \approx 42\%$), $\tilde{l} \to (2/\sqrt{3})\,GM/c$, and $L_{in} \to 2N\hbar$ [6].

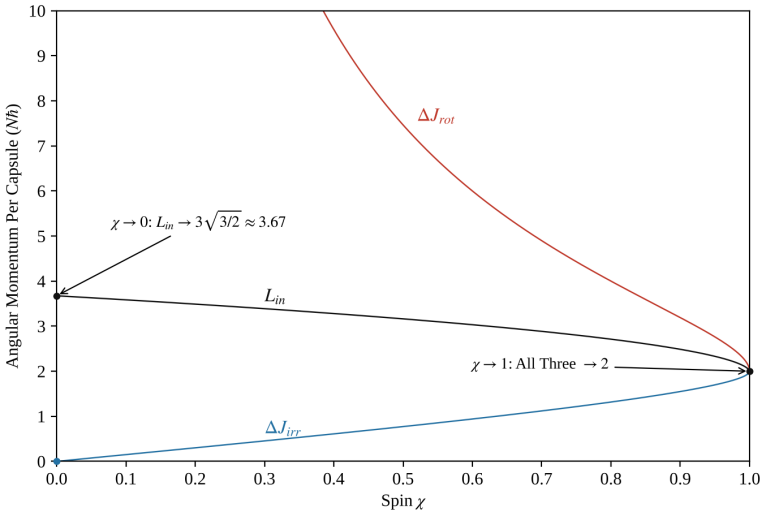

Fig. 3 The angular momentum L_{in} actually brought in by one capsule and the two registrations ΔJ_{rot}, ΔJ_{irr} as functions of spin χ (in units of $N\hbar$; vertical axis cut at 10). ω is obtained from $\chi = \sin 2\omega$ (Eq. (6)); ΔJ_{rot} and ΔJ_{irr} use Eqs. (48) and (49), and L_{in} uses Eqs. (50)–(53).

Figure 3 and the limits at the two ends show that over the whole range $0 < \chi < 1$

$$\Delta J_{irr} < L_{in} < \Delta J_{rot}. \tag{54}$$

The two ends: for $\chi \to 0$, $\omega \to 0$, $\Delta J_{irr} \to 0$, $\Delta J_{rot} \propto \cot\omega \to \infty$ and $L_{in} \to 3.67N\hbar$; for $\chi \to 1$, $\omega \to \pi/4$, $\Delta J_{rot} \to 2N\hbar\cot(\pi/4) = 2N\hbar$, $\Delta J_{irr} \to 2N\hbar\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\tan\frac{\pi}{8}\right) = 2N\hbar$, and $L_{in} \to 2N\hbar$: all three tend to $2N\hbar$ (shown in the figure as all tending to 2). So a capsule registered in the irreducible part always has a surplus $L_{in} - \Delta J_{irr}$, and one registered in the rotational part always has a shortfall $\Delta J_{rot} - L_{in}$. Since registration can only be done in whole packs, a difference between the angular momentum actually brought in and the angular momentum registered is unavoidable.

5.4 Probabilistic rule for local registration

Registration is a binary choice: each capsule is registered as a whole pack in either the irreducible part or the rotational part, and a capsule cannot be split between the two parts. The registration type is decided locally for each capsule, yet it must be correlated with the angular momentum the capsule brings in. The simplest rule

that conserves angular momentum on average is that each capsule is registered in the rotational part with probability

$$p = \frac{L_{in} - \Delta J_{irr}}{\Delta J_{rot} - \Delta J_{irr}} \tag{55}$$

and in the irreducible part with probability $1 - p$. By Eq. (54), $0 < p < 1$. In Fig. 3, p is the distance from L_{in} to the lower line ΔJ_{irr} as a fraction of the distance between the two lines, and $1 - p$ is the fraction for the distance from L_{in} to the upper line ΔJ_{rot}: the closer L_{in} is to a line, the higher the probability of registration in that class. The mean angular momentum registered per capsule is

$$\begin{aligned} \langle \Delta J \rangle &= p\,\Delta J_{rot} + (1-p)\,\Delta J_{irr} = \Delta J_{irr} + p\,(\Delta J_{rot} - \Delta J_{irr}) \\ &= \Delta J_{irr} + (L_{in} - \Delta J_{irr}) = L_{in}, \end{aligned} \tag{56}$$

exactly equal to the angular momentum brought in.

This allows the mean spin evolution to be checked. The mean angular momentum registered per capsule is L_{in}, so from Eq. (45):

$$\frac{d\chi}{d\ln M} = \frac{L_{in}}{N\hbar} - 2\chi = \frac{\tilde{l}c}{\tilde{e}GM} - 2\chi, \tag{57}$$

which is the same as the spin-evolution equation of thin-disk accretion in general relativity [5]. On average, the framework agrees with the established result; since Eq. (55) is itself constructed to conserve angular momentum on average, this shows that the probabilistic rule is compatible with the established evolution equation, not an independent test.

5.5 Differences and jets

The mean registered angular momentum equals the angular momentum brought in, but each capsule individually still has a difference. A capsule registered in the irreducible part (probability $1 - p$) has a surplus $L_{in} - \Delta J_{irr}$; one registered in the rotational part (probability p) has a shortfall $\Delta J_{rot} - L_{in}$. From Eq. (55),

$$1 - p = \frac{(\Delta J_{rot} - \Delta J_{irr}) - (L_{in} - \Delta J_{irr})}{\Delta J_{rot} - \Delta J_{irr}} = \frac{\Delta J_{rot} - L_{in}}{\Delta J_{rot} - \Delta J_{irr}}.$$

So the mean surplus per capsule equals the mean shortfall:

$$(1-p)\,(L_{in} - \Delta J_{irr}) = p\,(\Delta J_{rot} - L_{in}) = \frac{(L_{in} - \Delta J_{irr})\,(\Delta J_{rot} - L_{in})}{\Delta J_{rot} - \Delta J_{irr}} \equiv G. \tag{58}$$

G is the angular momentum exchanged in both directions between the black hole and its exterior for each pack absorbed. Before entering the horizon no pack has yet formed; matter crosses the horizon sub-step quantum by sub-step quantum: the angular momentum required by the registration is handed to the black hole as a whole

with the sub-step quanta, and a capsule carries no angular momentum of its own once registered as a pack; the surplus is handed to the exterior before crossing the horizon, in the plunging region between the ISCO and the horizon, held temporarily by fields or carried away by particles (the electromagnetic field in the magnetosphere, photons, charged particles in jets and so on); the shortfall is made up by the exterior (Section 5.1: the gravitational field, and matter and fields in the accretion disk and magnetosphere). Because each pack is registered locally without coordination, surpluses and shortfalls occur at different places and in different packs and cannot cancel on the spot. Figure 4 shows G and L_{in} as functions of χ: G/L_{in} decreases as χ increases, with $G \to L_{in}$ as $\chi \to 0$ and $G \to 0$ as $\chi \to 1$.

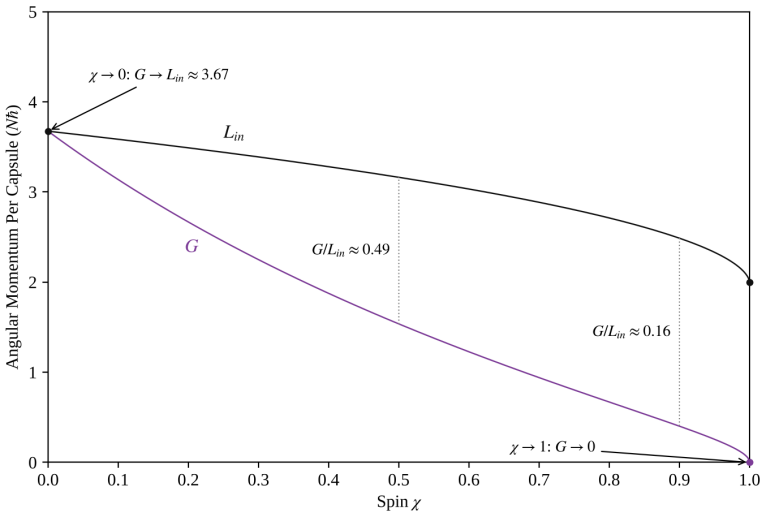

Fig. 4 The angular momentum L_{in} brought in per capsule and the two-way exchanged angular momentum G as functions of spin χ (in units of $N\hbar$). L_{in} uses Eqs. (50)–(53) and G uses Eq. (58); the dashed lines mark G/L_{in} at $\chi = 0.5$ and 0.9.

The surplus angular momentum must have an outlet. Its flux equals G per pack times the number of capsules absorbed per unit time, is proportional to $M\dot{M}$, and weakens as the spin rises: for a black hole that starts accreting from zero spin, with the spin rising gradually, even if the accretion rate grows with mass, the flux falls to about 1/3 of its initial value when the mass has grown by a factor 1.5 ($\chi \approx 0.84$), and to about 1/8 at a factor 2 ($\chi \approx 0.99$). Observationally, the fraction of radio-loud AGN rises steeply with black-hole mass [22, 23], and jet power increases with spin [24]; the former is consistent with a flux proportional to $M\dot{M}$, while the latter runs opposite to it. So G is not the main source of powerful jets, which come from the extraction of the black hole's rotational energy by the Blandford–Znajek process [25]. The effect of the registration difference on the spin is also small: suppose a fraction f_{out} of the surplus leaves and does not return; even for $f_{out} = 1$, the equilibrium spin is still about 0.9994 (for $f_{out} \leq 0.5$ there is no equilibrium and the spin keeps rising), which only makes

the spin rise slightly more slowly than in general relativity. In magnetohydrodynamic simulations, jets lower the equilibrium spin to about 0.93 [26], but those jets also extract the black hole's own rotational energy. What the framework adds is the part of the outflow that necessarily exists at low and intermediate spin; with sufficient magnetic field it can be collimated along the spin axis, and without a magnetic field it is held temporarily by the gravitational field and carried away by radiation.

5.6 Spin-down and re-registration in evaporation

From Section 5.2, accretion of positive energy does not decrease J; the registered J can decrease in three ways: by re-registration (N unchanged, the difference held temporarily by the field and carried off by radiation; Section 4.6), by reducing the number of capsules, or by breaking.

(1) Spin-down in evaporation. In semiclassical calculations, a rotating black hole radiates angular momentum several times faster than energy, and a rapidly rotating black hole spins down to near zero before losing most of its mass [8, 27]; in the summary of Taylor, Chambers and Hiscock [28], the spin is already close to zero when about half the mass has been lost. These semiclassical results apply at large N; for small N the framework reaches the same conclusion (Section 4.6). In this framework what is emitted is always an irreducible capsule; a rotational capsule must first be re-registered as an irreducible capsule and is then emitted. Following Page's results, angular momentum is lost first, which corresponds to frequent re-registration during emission, and the framework is compatible with this. Re-registered capsules are ordered by the temperature at registration; by last in, first out, the last pack registered on the outer horizon is the first to be emitted. The whole-pack counting inside is not visible from outside; only the sub-steps are. Each emitted pack is seen as $2N-1$ sub-step quanta, which together carry away energy E_P and angular momentum L, so the externally observable change of angular momentum proceeds in units of sub-steps (Section 3.6). Since L accumulates from the sub-steps and need not equal the whole-pack registered value exactly, the difference is held temporarily in the field, and on average this gives the probabilistic rule of Eq. (55). By Section 5.1, the angular-momentum difference is held temporarily by the field, so the emitted quanta need not carry away all the angular momentum of that capsule within one pack; the spin decreases continuously, consistent with Page's results. For comparison, if the angular momentum carried away by each emitted pack is taken to be the value L_{in} brought in by accretion (Eq. (53)), evaporation retraces the accretion path. From Eq. (57), integrating from the initial spin χ_0 to zero angular momentum,

$$\ln \frac{N}{N_0} = \int_{\chi_0=0.99}^{0} \frac{d\chi}{\tilde{l}c/(\tilde{e}GM) - 2\chi} \approx -0.708, \qquad \frac{N}{N_0} \approx e^{-0.708} \approx 0.49, \tag{59}$$

where $\tilde{l}$ and $\tilde{e}$ vary with χ according to Eqs. (50) and (51), and the integral is evaluated numerically. The value 0.49 is close to Page's result (about 0.50). As $\chi_0 \to 1$ the integral converges to $-\ln\sqrt{6}$ and $N/N_0 = 1/\sqrt{6} \approx 0.41$. This is exactly the reverse of accretion: spinning a non-rotating black hole up to the extremal state by thin-disk accretion requires the mass to grow by a factor $\sqrt{6}$, in agreement with Bardeen's

analytic result [5]. Along this path the proportion of emissions accompanied by re-registration is far above the share of rotational capsules in the black hole: about 2.4 times at the start, rising as the spin falls to more than a hundred times in the low-spin stage, so the angular momentum reaches zero before the mass does. Real evaporation is irreversible and follows Page's path.

(2) A rotational capsule is first re-registered, then emitted ($N_{rot} \to N_{irr}$). A rotational capsule that is first re-registered and then emitted as an irreducible capsule reaches the same final state as directly emitting a rotational pack: N_{irr} is unchanged, the outer-horizon area first increases and then decreases with no net change, and the angular momentum carried out is the registered value ΔJ_{rot} of a rotational pack. Its ratio to the energy: from Eq. (48), $\Delta J_{rot} \simeq 2N\hbar\cot\omega$; using $\hbar/E_P = \ell_p/c$,

$$\frac{\Delta J_{rot}}{E_P} = \frac{2N\hbar\cot\omega}{E_P} = \frac{2N\ell_p\cot\omega}{c}.$$

The horizon angular velocity Ω_H is the angular velocity with which the outer horizon rotates with the black hole; the Kerr solution gives $\Omega_H = ac/(r_+^2 + a^2)$ [29]. Using $r_+^2 + a^2 = R_s r_+$, $a = (R_s/2)\sin 2\omega$ and $r_+ = (R_s/2)(1+\cos 2\omega)$ (Paper II [10]):

$$\Omega_H = \frac{c\,(R_s/2)\sin 2\omega}{R_s\cdot(R_s/2)(1+\cos 2\omega)} = \frac{c\cdot 2\sin\omega\cos\omega}{R_s\cdot 2\cos^2\omega} = \frac{c\tan\omega}{R_s} = \frac{c\tan\omega}{2N\ell_p}, \tag{60}$$

so

$$\frac{\Delta J_{rot}}{E_P} = \frac{2N\ell_p\cot\omega}{c} = \frac{2N\ell_p}{c\tan\omega} = \frac{1}{\Omega_H}. \tag{61}$$

That is, when a rotational capsule is re-registered once and then emitted, the ratio of the angular momentum carried out to the energy is exactly $1/\Omega_H$.

Introducing the effective moment of inertia $I \equiv J/\Omega_H$, Eq. (42), Eq. (60) and $\tan\omega = k/N_{irr}$ give $I = 4\hbar N N_{irr}^2\ell_p/c$; with $\hbar/c = m_P\ell_p$, $I = 4NN_{irr}^2 m_P\ell_p^2 = MR_{A+}^2$, as for a thin ring of mass M at the outer-horizon areal radius R_{A+}. It is fixed by N and N_{irr} and is not a new variable. Eq. (61) can therefore be read as follows: for the same pack of energy E_P, the larger the moment of inertia, i.e. the smaller $\Omega_H = J/I$, the more angular momentum is registered. The angular momentum actually carried out is decided by the radiation and need not equal ΔJ_{rot}. Suppose that for each pack emitted the energy is E_P and the angular momentum actually carried out is L (taken positive), i.e. $\Delta(Mc^2) = -E_P$ and $\Delta J = -L$. From the first law $d(Mc^2) = T\,dS_+ + \Omega_H\,dJ$ (S_+ the entropy of the outer horizon, T the Hawking temperature):

$$\begin{aligned} T\,\Delta S_+ &\simeq \Delta(Mc^2) - \Omega_H\,\Delta J = -E_P + \Omega_H L \\ &= E_P\,\Omega_H\left(\frac{L}{E_P} - \frac{1}{\Omega_H}\right) = \Omega_H\,(L - \Delta J_{rot})\,, \end{aligned} \tag{62}$$

where the last step uses $E_P/\Omega_H = \Delta J_{rot}$ from Eq. (61). T and Ω_H are both positive, so the sign of ΔS_+ depends only on the difference between the angular momentum L actually carried out and the registered value ΔJ_{rot} of a rotational pack; S_+ is proportional to the outer-horizon area, and $S_+ = 4\pi k_B N_{irr}^2$. Each pack emitted is completed through sub-steps, and N is unchanged while the sub-step quanta are

being released. Eq. (62) takes representative values of T and Ω_H over one pack and is a large-N first-order approximation: one pack changes them by a relative amount $O(1/N)$, giving an error of $O(\hbar)$, whereas $L - \Delta J_{rot}$ is itself $O(N\hbar)$; so the sign of ΔS_+ is affected only for χ within $O(1/N)$ of χ_c, which is negligible at large N. A strict treatment would integrate $T\,dS_+ + \Omega_H\,dJ$ along the path. Near the upper limit the temperature is very low and its relative change within one pack becomes larger, so the approximation should not be extrapolated to the vicinity of the limit.

The number of re-registrations corresponding to each pack of energy emitted is denoted m; m is not decided in advance but results from re-registering pack by pack at the pace at which the angular-momentum difference is cleared. In the large-N first-order approximation each re-registration reduces the angular momentum by the same amount, so with m re-registrations during the emission of one pack the angular momentum carried out is $\Delta J_{irr} + m(\Delta J_{rot} - \Delta J_{irr})$; $m = 0$ and 1 give ΔJ_{irr} and ΔJ_{rot} respectively. Its average is set equal to L, i.e. $\Delta J_{irr} + \bar{m}(\Delta J_{rot} - \Delta J_{irr}) = L$. One re-registration increases the outer-horizon entropy by $\Delta S_{re} = 4\pi k_B(2N_{irr}+1) \simeq 8\pi k_B N_{irr}$; in a direct emission the outer-horizon entropy decreases by about the same amount, so Eq. (62) gives $\Delta J_{rot} - \Delta J_{irr} = T\Delta S_{re}/\Omega_H$. Further, from $\Delta J_{rot} = E_P/\Omega_H$ in Eq. (61) and from Eq. (8), with $N_{irr} = N\cos\omega$, $T\Delta S_{re} \simeq E_P \cos 2\omega/\cos\omega$. The average number of re-registrations is therefore

$$\bar{m} = \frac{L - \Delta J_{irr}}{\Delta J_{rot} - \Delta J_{irr}} = 1 + \frac{\Delta S_+}{\Delta S_{re}} = 1 + \Omega_H \left(\frac{L}{E_P} - \frac{1}{\Omega_H} \right) \frac{\cos\omega}{\cos 2\omega}, \tag{63}$$

where $\cos 2\omega = \sqrt{1-\chi^2}$. Eqs. (62) and (63) give one and the same criterion: the comparison of L/E_P with $1/\Omega_H$, i.e. of L with ΔJ_{rot}; ΔS_+ and $\bar{m} - 1$ have the same sign. There are three cases:

- $L < \Delta J_{rot}$: $\Delta S_+ < 0$, $\bar{m} < 1$, and the outer-horizon area decreases; fewer than one re-registration occurs during the emission of one pack, $\bar{m}$ is the probability of one re-registration, the same as the probabilistic rule of Section 5.4, and N_{irr} decreases.
- $L = \Delta J_{rot}$: $\Delta S_+ = 0$, $\bar{m} = 1$, and the outer-horizon area has no net change. This is exactly one re-registration followed by emission: the re-registration increases S_+ by $4\pi k_B(2N_{irr}+1)$ and the emission decreases it by the same amount.
- $L > \Delta J_{rot}$: beyond the range of one re-registration (carried away by superradiant quanta, or through the Penrose process, with negative energy and negative angular momentum flowing into the horizon), $\Delta S_+ > 0$, $\bar{m} > 1$, and the outer-horizon area increases on net, i.e. more than one re-registration occurs during the emission of one pack: the last pack re-registered is emitted first and the rest remain as irreducible capsules, so N_{irr} increases on net, consistent with Page's conclusion [8].

$1/\Omega_H$ is also the boundary of the superradiance condition $\varpi = m_\varphi \Omega_H$ (ϖ the angular frequency of the radiation, m_φ the angular-momentum quantum number along the axis): superradiant quanta have $m_\varphi\hbar/(\hbar\varpi) > 1/\Omega_H$, so $L/E_P > 1/\Omega_H$ means that the radiation lies on average on the superradiant side; superradiance dominates at rapid rotation [8, 30].

Quantitative comparison with Hawking radiation. Semiclassical calculations give the mass-loss rate and angular-momentum-loss rate of Hawking radiation from a rotating black hole, whose ratio varies with χ [8]. From these we obtain L_{out}, the angular momentum carried out by the radiation for each pack of energy E_P emitted. It is computed at each χ from the emission rates of the various particles, using the data of Dong, Kinney and Stojkovic [31], and is plotted as dashed lines in Fig. 5. Two limits are taken: Page's [8] photons, neutrinos and gravitons, applicable to low-temperature black holes that can emit only massless particles; and all Standard Model particles in the massless limit, applicable to high-temperature black holes with $k_B T_H \gg 100$ GeV. Real black holes lie between the two.

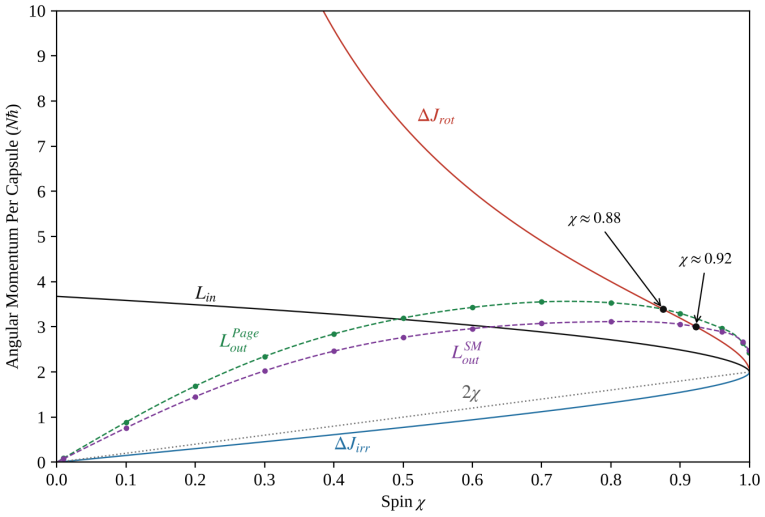

Fig. 5 Angular momentum brought in or carried out per capsule in accretion and evaporation (large-N first-order approximation, in units of $N\hbar$). The red and blue lines are the two registered values ΔJ_{rot} and ΔJ_{irr} (Eqs. (48) and (49)); the black line is the accretion L_{in} (Eq. (53)); the grey dotted line is the baseline 2χ (Eq. (45)); the dashed lines are the L_{out} carried out by Hawking radiation, green for Page's [8] photons, neutrinos and gravitons and purple for all Standard Model particles (massless limit). The black dots are the intersections χ_c where $L_{out} = \Delta J_{rot}$.

In Fig. 5 the grey dotted line is the baseline 2χ (Eq. (45)). The accretion L_{in} lies above it, i.e. Eq. (53), and the spin rises; the evaporation L_{out} also lies above it, and as mass decreases the spin falls, which is Page's conclusion. L_{out} always lies above the baseline 2χ, so in evaporation the spin only falls and the black hole evolves along the horizontal axis from right to left: for $\chi > \chi_c$ the outer-horizon entropy increases on average per emitted pack, and after falling below χ_c it decreases instead (Eq. (62)). As $\chi \to 0$, $L_{out} \to 0$: a non-rotating black hole emits no net angular momentum, in contrast with $L_{in} \to 3.67 N\hbar$ for accretion. At low and intermediate spin L_{out} lies between ΔJ_{irr} and ΔJ_{rot}, which is the case $\bar{m} < 1$ of Eq. (63), and recovers Page's spin-down on average. When the spin exceeds χ_c (about 0.88 for Page's set, about

0.92 for the Standard Model), $L_{out} > \Delta J_{rot}$: the angular momentum carried out per pack of energy emitted exceeds that carried out by one re-registration followed by emission. This crossing is insensitive to the particle species: as $\chi \to 1$, L_{out} in both limits tends to about 2.4–2.5 $N\hbar$, whereas $\Delta J_{rot} \to 2N\hbar$.

The crossing does not contradict the framework; it is exactly the case $L > \Delta J_{rot}$ above. From Eq. (62) with $L = L_{out}$, $T\,\Delta S_+ = \Omega_H(L_{out} - \Delta J_{rot})$. So, in the radiation process, the intersection of L_{out} and ΔJ_{rot} is exactly the boundary at which the outer-horizon entropy S_+ switches from increasing to decreasing, i.e. the location of the criterion $L_{out}/E_P = 1/\Omega_H$: for $\chi > \chi_c$, S_+ increases, and for $\chi < \chi_c$ emission decreases S_+, consistent with Page's finding that the area of a rapidly rotating black hole first increases. Page [8] gave only the qualitative conclusion (for sufficiently rapid rotation the area and entropy first increase); by the first law, this threshold is the χ_c solved for in this framework. The excess $L_{out} - \Delta J_{rot}$ comes from the additional re-registrations during the emission of one pack (N unchanged, N_{irr} increasing, $S_+ \propto N_{irr}^2$ increasing): during the emission of one irreducible pack more than one re-registration occurs; for Page's particle set, about 1.79 on average at $\chi = 0.99$. In this way the registration rules are compatible with Hawking radiation over the whole spin range; the value of χ_c depends on the particle species, while the meaning of the intersection is fixed by the first law.

From the L_{out} data in Fig. 5, Eq. (63) gives the balance ratio $\bar{m}$ between re-registration and emission at each spin, listed in Table 7. In the table, "Page" denotes photons, neutrinos and gravitons and "Standard Model" all Standard Model particles (massless limit); $\chi_c \approx 0.875$ for Page's set and $\chi_c \approx 0.923$ for the Standard Model. For $\bar{m} < 1$ the ratio re-registered : emitted is written $1 : (1/\bar{m})$, i.e. the number of packs emitted per pack re-registered. For $\chi < \chi_c$ fewer packs are re-registered than emitted: each pack re-registered can be paired with one pack emitted, i.e. re-register one and emit one, and the rest are emitted directly, so N_{irr} decreases gradually; at $\chi = \chi_c$ the two are equal; for $\chi > \chi_c$ more packs are re-registered than emitted, and at certain moments an extra pack must be re-registered to make up the angular-momentum difference; the extra re-registered capsules remain as irreducible capsules, N_{irr} increases and the outer-horizon area grows. As $\chi \to 1$, $\Delta J_{rot} - \Delta J_{irr}$ is very small and the ratio is very sensitive to L_{out}; the row $\chi = 0.99$ shows only the trend.

(3) The Penrose process and superradiance. In general relativity, both let particles carrying negative energy and negative angular momentum fall into the black hole, thereby reducing M and J [21, 30]. In this framework every capsule has energy $+E_P$, and there are no negative-energy capsules; the infall of negative-energy particles amounts to the black hole losing energy, of the same kind as evaporation, not accretion. When the energy lost is less than one pack E_P, the registration appears as re-registration (N unchanged); only when it accumulates to one pack E_P is one capsule emitted.

Table 7 Balance ratio between re-registration of rotational capsules and emission of irreducible capsules in evaporation (large-N first-order approximation)

χ	$\bar{m}$ (Page)	Re-reg. : emit (Page)	$\bar{m}$ (Standard Model)	Re-reg. : emit (Standard Model)
0.1	0.02	1 : 54	0.02	1 : 65
0.3	0.15	1 : 6.7	0.12	1 : 8.0
0.5	0.36	1 : 2.8	0.30	1 : 3.4
0.7	0.64	1 : 1.6	0.52	1 : 1.9
0.8	0.83	1 : 1.2	0.67	1 : 1.5
0.875	1.00	1 : 1	0.84	1 : 1.2
0.9	1.06	1.06 : 1	0.92	1 : 1.1
0.923	1.13	1.13 : 1	1.00	1 : 1
0.95	1.25	1.25 : 1	1.15	1.15 : 1
0.98	1.57	1.57 : 1	1.54	1.54 : 1
0.99	1.79	1.79 : 1	1.83	1.83 : 1

6 Breaking of large-N black holes

Section 4.5 discussed one-step breaking at small N. This section discusses large-N black holes: how the spin reaches the upper limit within the allowed range, and how breaking occurs at the limit.

6.1 Upper limit of the allowed range

The state whose horizon separation equals exactly the lower limit $2\ell_p$ has spin $\chi_{max} = 1 - 1/N^2$ from Eq. (7). It is the highest allowed spin (at small N, an envelope; see Section 3.8), not the extremal point; the classical extremal point is the state of zero separation, $\chi = 1$. From Eq. (41), at fixed N, $J \propto \sqrt{S_+ S_-}$, while $S_+ + S_- = 4\pi k_B N^2$ is fixed; the product is largest at $S_+ = S_-$, where the two horizons coincide ($\omega = \pi/4$), i.e. at the classical extremal point. So the maximum of angular momentum cannot be reached: breaking has already occurred before it, and the highest allowed spin is χ_{max}. The distance of χ_{max} from the extremal state is

$$1 - \chi_{max} = \frac{1}{N^2} = \left(\frac{2\ell_p}{R_s}\right)^2 \tag{64}$$

(using $R_s = 2N\ell_p$), i.e. the square of the ratio of the minimal horizon separation to the size of the black hole, which measures how strongly the criterion constrains that black hole. For astrophysical black holes it is always extremely small: for one solar mass, $N = M/m_P \approx 1.989 \times 10^{30}/2.176 \times 10^{-8} \approx 9.1 \times 10^{37}$ and $1/N^2 \approx 1.2 \times 10^{-76}$; for TON 618 (about 6.6×10^{10} solar masses), $N \approx 6.0 \times 10^{48}$ and $1/N^2 \approx 2.7 \times 10^{-98}$. So the spin limit of astrophysical black holes is set by accretion physics (Section 6.3). The χ_{max} of Eq. (7) is a continuous envelope. An integer state with exactly $g = 2\ell_p$ must satisfy $N_{irr} - k = 1$, i.e. $N^2 = N_{irr}^2 + (N_{irr} - 1)^2$, which is equivalent to the Pell equation

$$(2N_{irr} - 1)^2 - 2N^2 = -1, \tag{65}$$

This result introduces no new assumption: the $2d_m$ axiom gives the lower limit $2\ell_p$ on the horizon separation, i.e. $N_{irr} - k = 1$, and combined with energy integerization, $N^2 = N_{irr}^2 + k^2$, this yields Eq. (65). All positive integer solutions of Eq. (65) are

$$(2N_{irr} - 1) + N\sqrt{2} = \left(1 + \sqrt{2}\right)^{2j+1}, \qquad j = 0, 1, 2, \ldots \tag{66}$$

$j = 0$ gives $(N, N_{irr}) = (1, 1)$, a single background E_c, not a black hole; $j = 1$ gives the first nontrivial solution $(5, 4)$, which is exactly the minimal rotating state $BH(5, 4)$; then follow $(29, 21)$, $(169, 120)$, $(985, 697)$, $(5741, 4060)$, and so on. The solutions obey the recursion

$$N_{j+2} = 6N_{j+1} - N_j; \tag{67}$$

$2N_{irr} - 1$ obeys the same recursion, and the ratio of successive N tends to $3 + 2\sqrt{2} \approx 5.83$. So the integer states on the existence boundary are not only discrete but also follow a definite generating rule. Only the black-hole solutions of Eq. (65) reach the upper limit exactly; for every other N the highest rotating state has a separation larger than $2\ell_p$.

6.2 One-step breaking at the upper limit

From Eq. (5), $k^2 = N^2 - N_{irr}^2$, so to first order $\Delta k \simeq (N\,\Delta N - N_{irr}\,\Delta N_{irr})/k$; and from Eq. (34), $g = R_{A+} - R_{A-} = 2\ell_p(N_{irr} - k)$.

Rotational pack ($\Delta N = 1$, $\Delta N_{irr} = 0$): $R_{A+} = 2N_{irr}\ell_p$ is unchanged; $\Delta k \simeq N/k$, so

$$\Delta R_{A-} = 2\ell_p\,\Delta k \simeq 2\ell_p\,\frac{N}{k} = 2\ell_p\,\frac{N}{N\sin\omega} = \frac{2\ell_p}{\sin\omega}, \qquad \Delta g_{rot} \simeq -\frac{2\ell_p}{\sin\omega}. \tag{68}$$

Irreducible pack ($\Delta N = \Delta N_{irr} = 1$): $\Delta R_{A+} = 2\ell_p$; $\Delta k \simeq N_{rot}/k$, so

$$\begin{aligned} \Delta R_{A-} &\simeq 2\ell_p\,\frac{N_{rot}}{k} = 2\ell_p\,\frac{N(1-\cos\omega)}{N\sin\omega} = 2\ell_p\tan\frac{\omega}{2}, \\ \Delta g_{irr} &\simeq 2\ell_p\left(1 - \tan\frac{\omega}{2}\right) > 0. \end{aligned} \tag{69}$$

(For $\omega < \pi/4$, $\tan(\omega/2) < \tan(\pi/8) < 1$, so $\Delta g > 0$.)

Near the upper limit $\omega \to \pi/4$: $\sin(\pi/4) = \sqrt{2}/2$; by the half-angle formula $\tan(\pi/8) = (1 - \cos(\pi/4))/\sin(\pi/4) = (1 - \sqrt{2}/2)/(\sqrt{2}/2) = \sqrt{2} - 1 \approx 0.414$. Substituting into Eqs. (68) and (69):

$$\Delta g_{rot} \simeq -\frac{2\ell_p}{\sqrt{2}/2} = -2\sqrt{2}\,\ell_p \approx -2.83\,\ell_p, \tag{70}$$

$$\Delta g_{irr} \simeq 2\ell_p\left(1 - \left(\sqrt{2} - 1\right)\right) = 2\left(2 - \sqrt{2}\right)\ell_p \approx +1.17\,\ell_p. \tag{71}$$

So near the upper limit, absorbing one rotational pack shrinks the separation by about $2.83\ell_p$, more than the lower limit $2\ell_p$, and the separation cannot approach zero

gradually: a state at the upper limit that absorbs one more rotational pack falls below the lower limit, and breaking is completed in one step. Absorbing one irreducible pack increases the separation by about $1.17\ell_p$, moving away from breaking. The conclusion is the same for large N and small N (Section 4.5): breaking is always completed in one step by a single rotational pack, with no asymptotic approach to the extremal state. Hence a large-N state whose separation lies in the range

$$2\ell_p \le g \lesssim \left(2+2\sqrt{2}\right)\ell_p \approx 4.83\ell_p \tag{72}$$

breaks on absorbing one more rotational pack; for finite N the exact condition follows from Eq. (35): $N_{irr} - \sqrt{(N+1)^2 - N_{irr}^2} < 1$.

6.3 Known spin limits and the conditions for reaching breaking

Known spin limits. The highest spin an accreting black hole can reach depends on the accretion model (Table 8).

Table 8 Highest spins given by the accretion models and the upper limit of this framework

Source	Highest (equilibrium) spin χ
Standard thin disk, counting only the torque of captured disk radiation [7]	0.998
Magnetohydrodynamic simulations including magnetic fields and jets [26]	0.93
General relativity itself	1
This framework (Eq. (7))	$1 - 1/N^2$

The first entry is generally called the Thorne limit. Simulations of strongly magnetized (magnetically arrested) disks give even lower equilibrium spins, about 0–0.1 [32]. The upper limit of this framework comes from the separation of the two horizons being not smaller than $2\ell_p$; for astrophysical black holes it differs from 1 only by $1/N^2$ (Eq. (64)), negligibly small, and imposes no constraint on accretion. Below we take the Thorne limit as the starting point.

Mass growth required from $\chi_{Thorne} = 0.998$ to the upper limit. For astrophysical black holes the difference between $\chi_{max} = 1 - 1/N^2$ and 1 is negligible, and we take $\chi = 1$ in the calculation (the extremal point itself cannot be reached; it serves here only as a limit).

From Eq. (57), $d\ln M = d\chi/[\tilde{l}c/(\tilde{e}GM) - 2\chi]$; integrating,

$$\ln\frac{M}{M_{Thorne}} = \int_{0.998}^{1}\frac{d\chi}{\tilde{l}c/(\tilde{e}GM) - 2\chi} \approx 0.106, \qquad \frac{M}{M_{Thorne}} \approx e^{0.106} = 1.11, \tag{73}$$

where $\tilde{l}$ and $\tilde{e}$ vary with χ according to Eqs. (50) and (51), and the integral is evaluated numerically; as $\chi \to 1$ the denominator tends to zero, but the integral converges. That

is, the mass must increase by about 11% (pure accretion can approach the extremal state within a finite mass [5]).

As an ideal lower bound, if every pack is registered as a rotational pack ($p = 1$, irreducible mass unchanged [33]), then from $M = M_{irr}/\cos\omega$, $M_{Thorne} = M_{irr}/\cos\omega_{Thorne}$, where $\omega_{Thorne} = \frac{1}{2}\arcsin 0.998 = 0.7538$ and $\cos\omega_{Thorne} = 0.7290$; $M/M_{Thorne} = \cos\omega_{Thorne}/\cos(\pi/4) = 0.7290/0.7071 = 1.031$, i.e. an increase of only about 3.1%. In real accretion $p < 1$, and this is only an idealization.

The required mass growth is small in both cases, so the difficulty lies not in the amount of accretion but in two conditions:

- The angular momentum retained must be above the baseline. The accretion flow must carry net prograde angular momentum, and after subtracting the part carried off by jets (fraction f_{out}), the mean angular momentum retained per capsule must still exceed $2\chi N\hbar$ (Eq. (45)). By Section 5.5, for $f_{out} \leq 0.5$ this condition holds all the way to the upper limit; for $f_{out} = 1$ the spin stops at about 0.9994 and cannot reach the upper limit.
- The Thorne limit must be passed. Ordinary black holes accrete baryonic matter; retrograde photons in the disk radiation are captured more easily, and the spin equilibrates at about 0.998 [4, 7]; with magnetic fields and jets included, the equilibrium spin is lower. How cosmic egg cells pass these limits is left as an open question (Section 8).

Ordinary black holes therefore remain near these limits (such as the Thorne limit), becoming long-lived cosmic somatic cells; only cosmic egg cells that meet the conditions above can reach the upper limit and break.

6.4 Allocation of angular momentum at the breaking point and its association with the spin direction of celestial bodies

The black-hole state before breaking is a solution (Eq. (66)) of the Pell equation (Eq. (65)) with $N_{\mathrm{Pell}} = N$ (here N is the total number of capsules, not N_{irr}), whose two horizons are separated by exactly $2\ell_p$. Adding one rotating capsule triggers breaking, and the critical state just beyond it has $N_b = N_{\mathrm{Pell}} + 1$. This critical state corresponds to no existing black hole and is used only for bookkeeping.

At the breaking point, $\chi = 1$ and the total angular momentum of the critical state is $J = \chi N_b^2\hbar = N_b^2\hbar$, where N_b is the total number of capsules of the critical state. It is divided into a rotational part J_{rot}, carried by the rotational packs, and an irreducible part J_{irr}, carried by the irreducible packs. Quantities without a superscript refer to the critical state before breaking; quantities with the superscript H refer to the Hubble-sphere initial state after breaking. The derivation proceeds in four steps.

Step 1 (number shares). At the critical state the two horizon entropies are equal, $S_+ = S_-$ (Eqs. (99) and (100) of Paper II [10]). With $S_+ = 4\pi k_B N_{irr}^2$ and $S_- = 4\pi k_B(N_b^2 - N_{irr}^2)$, the condition $S_+ = S_-$ gives

$$N_{irr}^2 = N_b^2 - N_{irr}^2 \quad \Longrightarrow \quad \frac{N_{irr}}{N_b} = \frac{1}{\sqrt{2}}, \qquad \frac{N_{rot}}{N_b} = 1 - \frac{1}{\sqrt{2}}. \tag{74}$$

Step 2 (angular-momentum shares). Assume that the total angular momentum is spread uniformly over the N_b packs, so that each pack carries $J/N_b = N_b\hbar$. Then

$$|J_{rot}| = N_{rot} \cdot N_b\hbar = \Big(1 - \frac{1}{\sqrt{2}}\Big)N_b^2\hbar, \qquad |J_{irr}| = N_{irr} \cdot N_b\hbar = \frac{1}{\sqrt{2}}N_b^2\hbar, \tag{75}$$

and these satisfy $|J_{rot}| + |J_{irr}| = N_b^2\hbar$.

Step 3 (signed zero-sum after renormalization). After breaking, the critical state turns into the Hubble-sphere initial state: the inner horizon disappears and the outer horizon becomes the Hubble sphere, which releases the total angular momentum $N_b^2\hbar$. For a body that moves around a centre while also spinning, the total angular momentum is the sum of orbital and spin angular momentum. Inside the Hubble-sphere initial state no local celestial structure has yet formed, and there is no corresponding local orbital reference frame, so the orbital angular momentum is zero and only spin angular momentum remains. The rotation axis of the parent black hole can therefore serve as the common reference direction for the angular momentum inside the initial state.

The Hubble-sphere initial state must satisfy the constraint that its net angular momentum is zero. The overall rotation compatible with this constraint is quoted as $\sim 10^{-9}$ rad/yr for the shear-free rigid-rotation model [36] and as $\sim 10^{-13}$ rad/yr for a model estimate [37]. These two literature values, in rad/yr, are quoted only to compare orders of magnitude; they do not enter the derivation. Satisfying the constraint requires an output term $-N_b^2\hbar$ that is not given inside the Hubble sphere; its mechanism is not yet determined and is kept here only as a constraint. At the transition point the black-hole-state magnitudes are renormalized to the magnitudes of the Hubble-sphere initial state by the factors α and β:

$$|J_+^H| = \alpha|J_{rot}|, \qquad |J_-^H| = \beta|J_{irr}|. \tag{76}$$

The two parts have opposite signs, and this relation does not depend on the observer. Choosing the observing direction so that the rotational part spins clockwise, we denote it J_+^H and the irreducible part J_-^H, so that $J_+^H + J_-^H = 0$. Which part is clockwise depends on the observing position and is therefore arbitrary. Taking magnitudes gives the zero-sum condition

$$|J_+^H| - |J_-^H| = 0. \tag{77}$$

The released total equals the total angular momentum of the critical state, $N_b^2\hbar$, which gives the normalization

$$|J_+^H| + |J_-^H| = N_b^2\hbar. \tag{78}$$

Step 4 (solution). Adding and subtracting Eqs. (77) and (78) gives $2|J_+^H| = N_b^2\hbar$ and $2|J_-^H| = N_b^2\hbar$, so that

$$|J_+^H| = |J_-^H| = \tfrac{1}{2}N_b^2\hbar. \tag{79}$$

By Eq. (76), $\alpha = |J_+^H|/|J_{rot}|$ and $\beta = |J_-^H|/|J_{irr}|$. Using Eq. (75),

$$\alpha = \frac{\frac{1}{2}N_b^2\hbar}{(1-1/\sqrt{2})N_b^2\hbar} = \frac{1}{2-\sqrt{2}} = 1+\frac{1}{\sqrt{2}}, \qquad \beta = \frac{\frac{1}{2}N_b^2\hbar}{\frac{1}{\sqrt{2}}N_b^2\hbar} = \frac{1}{\sqrt{2}}. \tag{80}$$

Hence the ratio of the magnitudes of the black-hole-state parts is

$$\frac{|J_{rot}|}{|J_{irr}|} = \frac{\beta}{\alpha} = \sqrt{2}-1. \tag{81}$$

The two parts have opposite signs, and this relation does not depend on the observer; which part is clockwise depends on the observing position and is arbitrary.

6.5 Temperature at the upper limit

From Eq. (8), the Kerr temperature is

$$\begin{aligned} k_BT &= \frac{E_P}{8\pi N}\left(1-\tan^2\omega\right) = \frac{E_P}{8\pi N}\cdot\frac{\cos^2\omega-\sin^2\omega}{\cos^2\omega} \\ &= \frac{E_P}{8\pi N}\cdot\frac{(\cos\omega-\sin\omega)(\cos\omega+\sin\omega)}{\cos^2\omega}. \end{aligned} \tag{82}$$

At the upper limit $g = R_{A+} - R_{A-} = R_s\cos\omega - R_s\sin\omega = 2\ell_p$ with $R_s = 2N\ell_p$, so

$$\cos\omega - \sin\omega = \frac{2\ell_p}{2N\ell_p} = \frac{1}{N}.$$

Further, $(\cos\omega+\sin\omega)^2 = 2-(\cos\omega-\sin\omega)^2 = 2-1/N^2$. Writing $u = \sqrt{2N^2-1}$, we have $\cos\omega+\sin\omega = u/N$, $\cos\omega = (u+1)/(2N)$ and $\sin\omega = (u-1)/(2N)$, so $(\cos\omega-\sin\omega)(\cos\omega+\sin\omega) = u/N^2$ and $\cos^2\omega = (u+1)^2/(4N^2)$. Substituting into Eq. (82) gives the exact form

$$k_BT_{min} = \frac{E_P\,u}{2\pi N(u+1)^2}, \qquad u = \sqrt{2N^2-1}; \tag{83}$$

only integer states satisfying Eq. (65) reach it exactly, and all other integer states have higher temperatures. At large N, $u \approx \sqrt{2}N$, $\cos\omega+\sin\omega \approx \sqrt{2}$ and $\cos^2\omega \approx 1/2$, so

$$k_BT_{min} = \frac{E_P}{8\pi N}\cdot\frac{(\cos\omega-\sin\omega)(\cos\omega+\sin\omega)}{\cos^2\omega} \approx \frac{E_P}{8\pi N}\cdot\frac{(1/N)\cdot\sqrt{2}}{1/2} \approx \frac{\sqrt{2}\,E_P}{4\pi N^2}. \tag{84}$$

Before breaking, the temperature of the black hole does not fall to zero, consistent with the usual statement of the third law of thermodynamics [29, 34]. Kehle and Unger gave a counterexample, in the charged, non-rotating case, of an extremal black hole forming in finite time [35]; this framework treats only uncharged Kerr black holes, the temperature formula, Eq. (8), is derived from the counting of the framework

and the first law, and according to the existence criterion the black hole has already broken before its temperature falls to zero. If the existence criterion applies equally to the two horizons of a charged, non-rotating black hole, then $r_+ - r_- \geq 2\ell_p$ gives $k_B T_{min} = E_P/[2\pi(N+1)^2]$, also proportional to $1/N^2$; at large N, T_{min} tends to zero, compatible with the result of Kehle and Unger.

7 Conclusions

1. **Basis of the criterion.** The lower limit $2\ell_p$ on the separation of the two horizons is the minimum distinguishable span of the $2d_m$ axiom, and has the same origin as the coefficient $1/4$ of the Bekenstein–Hawking entropy.
2. **Angular-momentum spectrum.** The counting form of the criterion is $N_{irr} - k \geq 1$: black holes with $N \leq 4$ cannot rotate, the minimal rotating state is $BH(5,4)$, and the number of rotating states for each N is about $(1-1/\sqrt{2})N - 1/2$ (Eq. (18)). The J of this framework is derived from energy quantization and is in general not an integer multiple of $\hbar$ (Eq. (22)); only Pythagorean-triple states satisfy the standard integer rule, with $\mathcal{J}$ an integer multiple of 24. Externally only the sub-steps are seen: the angular momentum on sub-steps is $J = 2\hbar\sqrt{ab}$ (Eq. (24)), with quantization condition $\sqrt{ab} \in \mathbb{N}$ (Eq. (27)); it comes in units of $2\hbar$, the smallest reachable rotating value is $8\hbar$, and it is compatible with the standard rule; the whole-pack spectrum is internal counting.
3. **Small-N black holes.** $N = 5$ is both the first stable complete structure and the minimal rotating state; the step-by-step growth of small black holes is structurally analogous to Ostwald's rule of stages. A rotating state with $N \leq 7$ breaks in one step on absorbing one rotational pack, so growth must be ordered; in evaporation, a state whose separation is close to the lower limit must first be re-registered and then emitted, and every rotating state must lose all of its angular momentum before falling to $N = 4$, thereafter evaporating pack by pack as a non-rotating black hole.
4. **Registration and angular momentum.** A capsule is registered as a whole pack in either the irreducible part or the rotational part. The angular momentum of a black hole is proportional to the geometric mean of the entropies of the two horizons, $J = \hbar\sqrt{S_+ S_-}/(2\pi k_B)$ (Eq. (41)), equivalent to the known area-product formula $A_+ A_- = (8\pi G J/c^3)^2$; it is a property of the whole and cannot be apportioned to individual capsules. With $2\chi N\hbar$ as the baseline, the first-order angular momenta of rotational and irreducible packs lie above and below it respectively; the angular momentum L_{in} brought in by thin-disk accretion (Eq. (53)) always lies between the first-order angular momenta of rotational packs (Eq. (48)) and irreducible packs (Eq. (49)), and local probabilistic registration recovers Bardeen's spin-evolution equation on average; the pack-by-pack difference G is a two-way exchange between the black hole and its exterior, whose outflow weakens as the spin rises and is not the main source of powerful jets.
5. **Evaporation and re-registration.** Large-N black holes spin down continuously in evaporation, the spin being close to zero when about half the mass has been lost (Page). If each emitted pack carries away the angular momentum brought in by accretion, evaporation retraces the accretion path (Eq. (59)): for

an initial spin of 0.99 the angular momentum reaches zero when N has fallen to about 0.49, close to Page's result; the proportion of emissions accompanied by re-registration is far above the share of rotational capsules, so the angular momentum reaches zero before the mass does. By the first law, for each pack emitted $T\,\Delta S_+ \simeq \Omega_H(L - \Delta J_{rot})$ (Eq. (62)). What is emitted is always an irreducible capsule, and a rotational capsule must first be re-registered; re-registration increases the outer-horizon entropy and is a thermodynamic tendency, whose rate is controlled by the radiation. There is a single criterion, the comparison of L/E_P with $1/\Omega_H$, and the average number of re-registrations $\bar{m}$ is fixed by Eq. (63) (Table 7); when the angular momentum carried out by Hawking radiation exceeds ΔJ_{rot}, $\bar{m} > 1$, more than one re-registration occurs on average for each irreducible pack emitted, and the outer-horizon area increases on average; the threshold χ_c is about 0.88 for Page's particle set and about 0.92 with all Standard Model particles included, locating Page's qualitative conclusion. Re-registration is how the framework expresses this known phenomenon.

6. **Breaking of large-N black holes.** The maximum of angular momentum lies at the classical extremal point where the two horizons coincide and cannot be reached; the highest allowed spin is $\chi_{max} = 1 - 1/N^2$, with $1 - \chi_{max} = (2\ell_p/R_s)^2$, negligible for astrophysical black holes. Near the upper limit, absorbing one rotational pack shrinks the separation by about $2.83\ell_p$, more than the lower limit, so breaking is completed in one step; the temperature does not fall to zero before breaking, consistent with the third law of thermodynamics.
7. **Conditions for reaching breaking.** Going from the Thorne limit to the upper limit by thin-disk accretion requires a mass increase of only about 11% (ideal lower bound about 3.1%); the difficulty is that the angular momentum retained after subtracting jets must be above the baseline, and the spin limits given by the various accretion models must be passed. Ordinary black holes therefore remain near these limits, becoming long-lived cosmic somatic cells.

8 Open questions

(1) What determines the fraction f_{out} of the surplus angular momentum that leaves, and how is it related to the magnetic field of the accretion disk (Section 5.5)?
(2) Can the average number of re-registrations $\bar{m}$ in large-N evaporation be derived within the framework itself, without the semiclassical L_{out} (Section 5.6)?
(3) By what mechanism is energy released at breaking (Section 6)?
(4) Where does the net prograde angular momentum required by cosmic egg cells come from (Section 6)?
(5) How do cosmic egg cells pass the spin limits given by the various accretion models (the Thorne limit of about 0.998 and the lower magnetohydrodynamic equilibrium values) to reach χ_{max} (Section 6.3)?
(6) How is charge registered on the two horizons, and what are the existence criterion and temperature of charged black holes (Section 6.5)?

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- **Materials availability.** Not applicable.
- **Code availability.** The scripts used for the numerical integrations and figures are available from the author on reasonable request.
- **Author contribution.** G. Gu is the sole author and is responsible for the conception, derivations, analysis and writing of this work.

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